Differential Equations 2280
Shortened Sample Final Exam
Problem Numbers Match the Long Sample Final Exam
Tuesday, 30 April 2018, 7:30-10:00am, LCB 215

Instructions: This in-class exam is 120 minutes. About 20 minutes per sub-section. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

The actual final exam will have 25 sections to solve. This sample exam has 50 sections to solve. It is intended as a study guide for the final exam, which is why it is twice as long as the actual final exam.

Chapters 1 and 2: Linear First Order Differential Equations

3. (Solve a Separable Equation)
Given $y^2y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.
(a) Find all equilibrium solutions.
(b) Find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly.

Answer:
(a) $y = 5/4$
(b)

$$-\frac{1}{3}\ln|125 - 64y^3| = 2x + \frac{3}{2}\ln(1+x^2) - 2\arctan(x) + c$$

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4. (Linear Equations)
(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.
(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.
(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p.

Answer:
(a) $v(t) = -24t - 8$
(b) $y(x) = Ce^{\sqrt{x+2}}$

Chapter 3: Linear Equations of Higher Order

6. (ch3)

(a) Solve for the general solutions:

(**a.1**) [25%] $y'' + 4y' + 4y = 0$,

(**a.2**) [25%] $\frac{d^6y}{dx^6} + 4\frac{d^4y}{dx^4} = 0$,

(**a.3**) [25%] Char. eq. $r(r-3)(r^3-9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

Answer:

(a)

1: $r^2 + 4r + 4 = 0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

2: $r^6 + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = c_1e^{0x} + c_2xe^{0x} + c_3\cos 2x + c_4 \sin 2x$.

3: Write as $r^3(r-3)^3(r+3)^2(r^2+4)^3 = 0$. Then y is a linear combination of the atoms 1, x, x^2, e^{3x}, xe^{3x}, x^2e^{3x}, e^{-3x}, xe^{-3x}, $\cos 2x$, $x\cos 2x$, $x^2\cos 2x$, $\sin 2x$, $x\sin 2x$, $x^2 \sin 2x$.

Part (b)

Use $6r^2 + 7r + 2 = 0$ and the quadratic formula to obtain roots $r = -1/2, -2/3$. Then $x(t) = c_1e^{-t/2} + c_2e^{-2t/3}$. This is over-damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

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7. (ch3)

Determine for $\frac{d^6y}{dx^6} + 4\frac{d^4y}{dx^4} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Answer:

The homogeneous solution is a linear combination of the atoms 1, x, x^2, x^3, $\cos x$, $\sin x$ because the characteristic polynomial has roots 0, 0, 0, 0, i, $-i$.

$\boxed{1}$ An initial trial solution y is constructed by Rule I for atoms 1, x, e^{3x}, e^{-3x}, $\cos x$, $\sin x$ giving

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 + d_2x + d_3x^2 + d_4x^3, \\ y_2 &= d_5\cos x + d_6 x\cos x, \\ y_3 &= d_7 \sin x + d_8 x \sin x, \\ y_4 &= d_9 e^{-x}. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.

$\boxed{2}$ Rule II is applied individually to each of y_1, y_2, y_3, y_4.

The result is the **shortest trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1 x^4 + d_2 x^5 + d_3 x^6 + d_4 x^7, \\ y_2 &= d_5 x \cos x + d_6 x^2 \cos x, \\ y_3 &= d_7 x \sin x + d_8 x^2 \sin x, \\ y_4 &= d_9 e^{-x}. \end{aligned}$$

Chapters 4 and 5: Systems of Differential Equations

9. (ch5)

The eigenanalysis method says that the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $\vec{\mathbf{x}}(t) = c_1 \vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t} + c_3 \vec{\mathbf{v}}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \vec{\mathbf{v}}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for A.

(b) [25%] Display the solution $\vec{\mathbf{x}}(t)$ of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

Answer:

(1): The eigenpairs are

$$\left(4, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

An expected detail is the cofactor expansion of $\det(A - \lambda I)$ and factoring to find eigenvalues $4, 6, 7$. Eigenvectors should be found by a sequence of swap, combo, mult operations on the augmented matrix, followed by taking the partial ∂_{t_1} on invented symbol t_1 in the general solution to compute the eigenvector. In short, the eigenvectors are Strang's Special Solutions, and in general there can be many eigenvectors for a single eigenvalue.

(2): The eigenanalysis method for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ implies

$$\vec{\mathbf{x}}(t) = c_1 e^{4t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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10. (ch5)

(a) [20%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) [40%] Putzer's Method removed from the final exam.

(c) [40%] Display the general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ according to the Cayley-Hamilton-Ziebur Method. In particular, display the equations that determine the three vectors in the general solution. **To save time**, don't solve for the three vectors in the formula.

(d) [40%] Display the general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ according to the Eigenanalysis Method. **To save time**, find one eigenpair explicitly, just to show how it is done, but don't solve for the last two eigenpairs.

(e) [40%] Display the general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ according to Laplace's Method. **To save time**, use symbols for partial fraction constants and leave the symbols unevaluated.

Answer:

(a) **Eigenvalue Calculation**

Subtract λ from the diagonal elements of A to obtain matrix $B = A - \lambda I$, then expand $\det(B)$ by cofactors to obtain the characteristic polynomial. The roots are the eigenvalues $\lambda = 2, 3, 5$.

(c) **Cayley-Hamilton-Ziebur Method**

The eigenvalues $2, 3, 5$ from (a) are used to create the list of atoms e^{2t}, e^{3t}, e^{5t}. Then the Cayley-Hamilton-Ziebur method implies there are constant vectors $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, $\vec{\mathbf{d}}_3$ which depend on $\vec{\mathbf{u}}(0)$ and A such that

$$\vec{\mathbf{u}}(t) = e^{2t}\vec{\mathbf{d}}_1 + e^{3t}\vec{\mathbf{d}}_2 + e^{5t}\vec{\mathbf{d}}_3.$$

It is known for the case of distinct eigenvalues that vectors $\vec{\mathbf{d}}_j$ are eigenvectors of A multiplied by arbitrary constants c_1, c_2, c_3, respectively. Discussed below is how to solve for the unknown vectors without eigenanalysis.

The determining equations are formed from differentiation of this formula two times, then replace $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}' = AA\vec{\mathbf{u}}$. Finally, remove t from the three equations by setting $t = 0$, and define $\vec{\mathbf{u}}_0 = \vec{\mathbf{u}}(0)$. Then the three equations are, with $\vec{\mathbf{u}}_0 = \vec{\mathbf{u}}(0)$,

$$\begin{array}{rcrcrcr} \vec{\mathbf{u}}_0 & = & \vec{\mathbf{d}}_1 & + & \vec{\mathbf{d}}_2 & + & \vec{\mathbf{d}}_3 \\ A\vec{\mathbf{u}}_0 & = & 2\vec{\mathbf{d}}_1 & + & 3\vec{\mathbf{d}}_2 & + & 5\vec{\mathbf{d}}_3 \\ A^2\vec{\mathbf{u}}_0 & = & 4\vec{\mathbf{d}}_1 & + & 9\vec{\mathbf{d}}_2 & + & 25\vec{\mathbf{d}}_3 \end{array}$$

This ends the solution to the problem. We continue just to illustrate how the unknown vectors are found directly, without eigenanalysis. The matrix of coefficients

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{pmatrix}$$

and its transpose matrix $B = C^T$ give a formal relation

$$\langle \vec{\mathbf{u}}_0 | A\vec{\mathbf{u}}_0 | A^2\vec{\mathbf{u}}_0 \rangle = \langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 | \vec{\mathbf{d}}_3 \rangle B.$$

Multiplying this relation by B^{-1} gives

$$\langle \vec{\mathbf{d}}_1 | \vec{\mathbf{d}}_2 | \vec{\mathbf{d}}_3 \rangle = \langle \vec{\mathbf{u}}_0 | A\vec{\mathbf{u}}_0 | A^2\vec{\mathbf{u}}_0 \rangle B^{-1}.$$

Then disassembling the formal matrix multiply implies

$$\begin{array}{rcrcrcr} \vec{\mathbf{d}}_1 & = & 5\vec{\mathbf{u}}_0 & - & \frac{8}{3}A\vec{\mathbf{u}}_0 & + & \frac{1}{3}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{d}}_2 & = & -5\vec{\mathbf{u}}_0 & + & \frac{7}{2}A\vec{\mathbf{u}}_0 & - & \frac{1}{2}A^2\vec{\mathbf{u}}_0 \\ \vec{\mathbf{d}}_3 & = & 5\vec{\mathbf{u}}_0 & - & \frac{5}{6}A\vec{\mathbf{u}}_0 & + & \frac{1}{6}A^2\vec{\mathbf{u}}_0 \end{array}$$

The matrix of coefficients is

$$\begin{pmatrix} 5 & -\frac{8}{3} & \frac{1}{3} \\ -5 & \frac{7}{2} & -\frac{1}{2} \\ 1 & -\frac{5}{6} & \frac{1}{6} \end{pmatrix} = \left(B^{-1}\right)^T = C^{-1}!$$

This fact, that solving for $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$, $\vec{\mathbf{d}}_3$ in the displayed equations reduces to inverting the matrix of coefficients, can be used as a shortcut in the Cayley-Hamilton-Ziebur method.

(d) **Eigenanalysis Method**

For matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

the eigenpairs are computed to be

$$\left(2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right).$$

Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ has general solution

$$\vec{\mathbf{u}}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(e) **Laplace's Method**

The start is the Laplace resolvent formula for matrix differential equation $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

$$(sI - A)\,\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}_0.$$

This formula expands to

$$\begin{pmatrix} s-4 & -1 & 1 \\ -1 & s-4 & 2 \\ 0 & 0 & s-2 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where symbols a, b, c are arbitrary constants for the initial data $\vec{\mathbf{u}}_0$. Let W denote the coefficient matrix. Then the inverse of W can be computed using the adjugate formula $W^{-1} = \mathbf{adj}(W)/\det(W)$. The answer for the inverse is

$$W^{-1} = \frac{1}{(s-5)(s-2)(s-3)} \begin{pmatrix} s^2-6\,s+8 & s-2 & -s+2 \\ s-2 & s^2-6\,s+8 & -2\,s+7 \\ 0 & 0 & s^2-8\,s+15 \end{pmatrix}$$

True, this formula can be derived and then followed by inverse Laplace methods to obtain an answer in variable t. However, we already know the outcome, because this matrix is the Laplace of the exponential matrix e^{At}. The exponential matrix formula was already derived in (b) above. Expanding the matrix multiplies and collecting terms gives the final answer

$$W^{-1} = \mathcal{L}\left(e^{At}\right) = \frac{1}{2}\mathcal{L}\begin{pmatrix} e^{5\,t}+e^{3\,t} & e^{5\,t}-e^{3\,t} & -e^{5\,t}+e^{3\,t} \\ e^{5\,t}-e^{3\,t} & e^{5\,t}+e^{3\,t} & -e^{5\,t}+2\,e^{2\,t}-e^{3\,t} \\ 0 & 0 & 2\,e^{2\,t} \end{pmatrix}$$

Canceling the $\mathcal{L}$ with Lerch's Theorem implies the same answer as found in part (b), which is

$$\vec{\mathbf{u}}(t) = e^{At}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5\,t}+e^{3\,t} & e^{5\,t}-e^{3\,t} & -e^{5\,t}+e^{3\,t} \\ e^{5\,t}-e^{3\,t} & e^{5\,t}+e^{3\,t} & -e^{5\,t}+2\,e^{2\,t}-e^{3\,t} \\ 0 & 0 & 2\,e^{2\,t} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

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11. (ch5) Do enough to make 100%

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Display the general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$. Show details from either the eigenanalysis method or the Laplace method.

(b) [50%] Using the same matrix A from part (a), display the solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ according to the Cayley-Hamilton-Ziebur Method. To save time, write out the system to be solved for the two vectors, and then stop, without solving for the vectors.

(c) [50%] Using the same matrix A from part (a), compute the exponential matrix e^{At} by any known method, for example, the formula $e^{At} = \Phi(t)\Phi^{-1}(0)$ where $\Phi(t)$ is any fundamental matrix, or via Putzer's 2×2 formula.

Answer:

(a) Eigenanalysis method

The eigenpairs of A are

$$\left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \quad \left(6, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

which implies the eigenanalysis general solution

$$\vec{\mathbf{u}}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b)Cayley-Hamilton-Ziebur method
Then $\vec{\mathbf{u}}(t) = e^{4t}\vec{\mathbf{c}}_1 + e^{6t}\vec{\mathbf{c}}_2$ for some constant vectors $\vec{\mathbf{c}}_1$, $\vec{\mathbf{c}}_2$ that depend on $\vec{\mathbf{u}}(0)$ and A. Differentiate this equation once and use $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, then set $t = 0$. The resulting system is

$$\begin{aligned} \vec{\mathbf{u}}_0 &= e^0\vec{\mathbf{c}}_1 + e^0\vec{\mathbf{c}}_2 \\ A\vec{\mathbf{u}}_0 &= 4e^0\vec{\mathbf{c}}_1 + 6e^0\vec{\mathbf{c}}_2 \end{aligned}$$

The **CHZ Shortcut** writes $x(t) = c_1e^{4t} + c_2e^{6t}$, then solve the first differential equation $x' = 5x + y$ for $y = x' - 5x$ and substitute the expression for $x(t)$ to obtain $y = 4c_1e^{4t} + 6c_2e^{6t} - 5c_1e^{4t} - 5c_2e^{6t} = -c_1e^{4t} + c_2e^{6t}$.
(c) Putzer Method (not required, course enrichment)
The result is $e^{At} = e^{4t}I + \frac{e^{4t}-e^{6t}}{4-6}(A - 4I)$. Functions r_1, r_2 are computed from $r_1' = 4r_1$, $r_1(0) = 1$, $r_2' = 6r_2 + r_1$, $r_2(0) = 0$.

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{4t} + e^{6t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{4t} + e^{6t} \end{pmatrix}.$$

Alternatively, the Putzer formula can be memorized for 2×2 matrices, which shortens the details considerably:

$$e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I) = \frac{1}{2}\begin{pmatrix} e^{4t} + e^{6t} & e^{6t} - e^{4t} \\ e^{6t} - e^{4t} & e^{4t} + e^{6t} \end{pmatrix}.$$

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12. (ch5) Do both

(a) [50%] Display the solution of $\vec{\mathbf{u}}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\vec{\mathbf{u}}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, using any method that applies. The expected answer is $\Phi(t)\vec{\mathbf{u}}(0)$, where Φ is a fundamental matrix.

(b) [50%] Display the variation of parameters formula for the system below. Then integrate to find $\vec{\mathbf{u}}_p(t)$ for $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$.

$$\vec{\mathbf{u}}' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\vec{\mathbf{u}} + \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}.$$

Answer:
(a) **Resolvent method**
The resolvent equation $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0)$ is the system

$$\begin{pmatrix} s-2 & 0 \\ -1 & s-2 \end{pmatrix}\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is solved by Cramer's rule for unknowns $\mathcal{L}(x)$, $\mathcal{L}(y)$ to obtain

$$\mathcal{L}(x) = \frac{0}{(s-2)^2}, \quad \mathcal{L}(y) = \frac{s-2}{(s-2)^2}.$$

The backward Laplace table implies

$$x(t) = 0, \quad y(t) = e^{2t}.$$

Eigenanalysis Method. It fails to apply because the matrix is not diagonalizable (has only one eigenpair, not two).

Integrating Factor method, Ch1. Look at the equations as scalar equations $x' = 2x$, $x(0) = 0$ and $y' = x + 2y$, $y(0) = 1$. Picard says $x(t) = 0$ and then $y' = 0 + 2y$, $y(0) = 1$ implies $y(t) = e^{2t}$ by the homogeneous equation shortcut, Ch1.

Cayley-Hamilton-Ziebur Shortcut. The eigenvalues are $2, 2$ and the atoms are e^{2t}, te^{2t}. Write $y = c_1e^{2t} + c_2te^{2t}$. Use the second differential equation $y' = x + 2y$ to solve for $x = y' - 2y = 2c_1e^{2t} + 2tc_2e^{2t} + c_2e^{2t} - 2c_1e^{2t} - 2c_2te^{2t}$. Reduce it to $x = c_2e^{2t}$. Now use $x(0) = 0$, $y(0) = 1$ to determine $c_2 = 0$, $c_2 = 1$. Then $x = 0, y = e^{2t}$.

Fundamental Matrix. From the general solution, take partials on symbols c_1, c_2 to find $\Phi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$. Then $e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}$.

Putzer Method

Putzer's exponential formula gives

$$e^{At} = e^{2t}I + te^{2t}(A - 2I) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}.$$

(b) **Variation of Parameters**

Then regardless of how we found e^{At}, the variation of parameters formula implies $\vec{\mathbf{u}}_p(t) = e^{At} \int_0^t e^{-Au} \begin{pmatrix} e^{2u} \\ 0 \end{pmatrix} du = e^{At} \int_0^t \begin{pmatrix} 1 \\ -u \end{pmatrix} du = \begin{pmatrix} te^{2t} \\ t^2e^{2t}/2 \end{pmatrix}$.

Chapter 6: Dynamical Systems

14. (ch6)

Find the equilibrium points of $x' = 14x - x^2/2 - xy$, $y' = 16y - y^2/2 - xy$ and classify each linearization at an equilibrium as a node, spiral, center, saddle. What classifications can be deduced for the nonlinear system, according to the **Paste Theorem**, which is textbook Theorem 2, section 6.2?

Answer:

The equilibria are constant solutions, which are found from the equations

$$\begin{aligned} 0 &= (14 - x/2 - y)x \\ 0 &= (16 - y/2 - x)y \end{aligned}$$

Considering when a zero factor can occur leads to the four equilibria $(0,0)$, $(0,32)$, $(28,0)$, $(12,8)$. The last equilibrium comes from solving the system of equations

$$\begin{aligned} x/2 + y &= 14 \\ x + y/2 &= 16 \end{aligned}$$

Linearization

The Jacobian matrix J is the augmented matrix of column vector partial derivatives $\partial_x \vec{\mathbf{F}}$, $\partial_y \vec{\mathbf{F}}$ computed from vector function

$$\vec{\mathbf{f}}(x,y) = \begin{pmatrix} 14x - x^2/2 - yx \\ 16y - y^2/2 - xy \end{pmatrix}.$$

Then

$$J(x,y) = \begin{pmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{pmatrix}.$$

The four matrices below are $J(x,y)$ when (x,y) is replaced by each of the four equilibrium points. Included in the table are the roots of the characteristic equation for each matrix and its classification based on the roots. No book was consulted for the classifications. The idea in each is to examine the limits at $t = \pm\infty$, then eliminate classifications. No matrix has complex eigenvalues, and that eliminates the center and spiral. The first three are stable at either $t = \infty$ or $t = \infty$, which eliminates the saddle and leaves the node as the only possible classification.

$$\begin{array}{lllll}
A_1 = J(0,0) = \begin{pmatrix} 14 & 0 \\ 0 & 16 \end{pmatrix} & r = 14, 16 & \text{node} \\
A_2 = J(0,32) = \begin{pmatrix} -18 & 0 \\ -32 & -16 \end{pmatrix} & r = -18, -16 & \text{node} \\
A_3 = J(28,0) = \begin{pmatrix} -14 & -28 \\ 0 & -12 \end{pmatrix} & r = -14, -12 & \text{node} \\
A_4 = J(12,8) = \begin{pmatrix} -6 & -12 \\ -8 & -4 \end{pmatrix} & r = -5 + \sqrt{97}, -5 - \sqrt{97} & \text{saddle}
\end{array}$$

Some maple code for checking the answers:

```
F:=unapply([14*x-x^2/2-y*x , 16*y-y^2/2 -x*y],(x,y));
Fx:=unapply(map(u->diff(u,x),F(x,y)),(x,y));
Fy:=unapply(map(u->diff(u,y),F(x,y)),(x,y));
Fx(0,0);Fy(0,0);Fx(28,0);Fy(28,0);Fx(0,32);Fy(0,32);Fx(0,32);Fy(0,32);
```

15. (ch6) Do enough to make 100%
(a) [25%] Which of the four types *center, spiral, node, saddle* can be unstable at $t = \infty$? Explain your answer.
(b) [25%] Give an example of a linear 2-dimensional system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ with a saddle

at equilibrium point $x = y = 0$, and A is not triangular.
(c) [25%] Give an example of a nonlinear 2-dimensional predator-prey system with exactly four equilibria.
(d) [25%] Display a formula for the general solution of the equation $\vec{\mathbf{u}}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{\mathbf{u}}$. Then explain why the system has a spiral at $(0, 0)$.
(e) [25%] Is the origin an isolated equilibrium point of the linear system $\vec{\mathbf{u}}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{\mathbf{u}}$? Explain your answer.

Answer:
(a) All except the center, which is stable but not asymptotically stable. All the others correspond to a general solution which can have an exponential factor e^{kt} in each term. If $k > 0$, then the solution cannot approach the origin at $t = \infty$.
(b) Required are characteristic roots like $1, -1$. Let $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Define $A = PBP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ has a saddle at the origin, because the characteristic roots of A are still $1, -1$. And $A = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$ is not triangular.
(c) **Example**: The nonlinear predator-prey system $x' = (x+y-4)x$, $y' = (-x+2y-2)y$ has exactly four equilibrium points $(0, 0)$, $(4, 0)$, $(0, 1)$, $(2, 2)$.
(d) The characteristic equation $\det(A - \lambda I) = 0$ is $(1 - \lambda)^2 + 1 = 0$ with complex roots $1 \pm i$ and corresponding atoms $e^t \cos t$, $e^t \sin t$. Then the Cayley-Hamilton-Ziebur Method implies

$$\vec{\mathbf{u}}(t) = e^t \cos t \vec{\mathbf{c}}_1 + e^t \sin t \vec{\mathbf{c}}_2.$$

Explanation, why the classification is a spiral. Such solutions containing sine and cosine factors wrap around the origin. This makes it a spiral or a center. Because of the exponential factor e^t, it is asymptotically stable at $t = -\infty$, which disallows a center, so it is a spiral.
(e) No, because $\det(A) = 0$. In this case, $A\vec{\mathbf{u}} = \vec{\mathbf{0}}$ has infinitely many solutions, describing a line of equilibria through the origin. This implies the equilibrium point $(0, 0)$ is not isolated [you cannot draw a circle about $(0, 0)$ which contains no other equilibrium point].

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16. (ch7)

(a) Define the direct Laplace Transform.

(b) Define Heaviside's unit step function.

(c) Derive a Laplace integral formula for Heaviside's unit step function.

(d) Explain all the steps in Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$.

Answer:
(a) Definition of Direct Laplace Transform

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt.$$

(b) Definition of the Heaviside unit step

$$u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$$

(c) Derivation
We prove the second shifting theorem $\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t))$, which includes an integral formula for the Heaviside function by substitution of $f(t) = 1$.

$$\begin{array}{rcl} \mathcal{L}(u(t-a)f(t-a)) & = & \int_0^\infty u(t-a)f(t-a)e^{-st}dt \\ & = & \int_0^a (\text{integrand})dt + \int_a^\infty (\text{integrand})dt \\ & = & 0 + \int_a^\infty f(t-a)e^{-st}dt \\ & = & \int_0^\infty f(u)e^{-s(a+u)}du \\ & = & e^{-sa}\int_0^\infty f(u)e^{-su}du \\ & = & e^{-as}\mathcal{L}(f(t)) \end{array}$$

Used in the derivation is a change of variable $u = t-a$, $du = dt$. Line 3 uses $u(t-a) = 0$ on the interval $0 \le t \le a$ and $u(t-a) = 1$ on $a \le t < \infty$, which simplifies each integrand. Line 5 observes that factor e^{-sa} in the integrand is a constant relative to u-integration, therefore it can move through the integral sign.
(d) Laplace's method explained.
The first step transforms the equation using the parts formula and initial data to get

$$(s+2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s-1)}{s+2} = \frac{s}{(s-1)(s+2)}.$$

Partial fraction methods imply

$$\mathcal{L}(x) = \frac{a}{s-1} + \frac{b}{s+2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

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17. (ch7)

(a) Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$.

(b) Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s-3)}$.

(c) Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(d) Find $\mathcal{L}(f)$ where $f(t)$ is the periodic function of period 2 equal to $t/2$ on $0 \le t \le 2$ (sawtooth wave).

Answer:
(a) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4}) = \frac{100}{3}\mathcal{L}(\sin t - \frac{1}{2}\sin 2t)$ implies $f(t) = \frac{100}{3}(\sin t - \frac{1}{2}\sin 2t)$.
(b) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-3} = \mathcal{L}(a + bt + ce^{3t})$ implies $f(t) = a + bt + ce^{3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = -1/3$, $c = 1/9$.
(c) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \frac{d}{ds}\left(\frac{1}{(s-2)^2+9}\right) = \frac{-2(s-2)}{((s-2)^2+9)^2}$.

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18. (ch7)

(a) Solve $y'' + 4y' + 4y = t^2$, $y(0) = y'(0) = 0$ by Laplace's Method.

(b) Solve $x''' + x'' - 6x' = 0$, $x(0) = x'(0) = 0$, $x''(0) = 1$ by Laplace's Method.

(c) Solve the system $x' = x + y$, $y' = x - y + e^t$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:
(a) Transform to get $\mathcal{L}(x) = \frac{\mathcal{L}(t^2)}{s^2+4s+4}$. Then $\mathcal{L}(x) = \frac{1}{s^3(s+2)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{s+2} + \frac{f}{(s+2)^2} = \mathcal{L}(a + bt + ct^2 + de^{-2t} + fte^{-2t})$. The answer is $x(t) = a + bt + +ct^2 + de^{-2t} + fte^{-2t}$. The partial fraction constants are $a = 3/16$, $b = -1/4$, $c = 1/4$, $d = -3/16$, $f = -1/8$.
(b) Transform to get $\mathcal{L}(x) = \frac{1}{s^3+s^2-6s} = \frac{1}{s(s-2)(s+3)} = \frac{a}{s} + \frac{b}{s-2} + \frac{c}{s+3} = \mathcal{L}(a + be^{2t} + ce^{-3t})$. Then the answer is $x(t) = a + be^{2t} + ce^{-3t}$. The partial fraction constants are $a = -1/6, b = 1/10, c = 1/15$.

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19. (ch7)
(a) [25%] Solve by Laplace's method $x'' + x = \cos t$, $x(0) = x'(0) = 0$.

(b) [10%] Does there exist $f(t)$ of exponential order such that $\mathcal{L}(f(t)) = \dfrac{s}{s+1}$? Details required.

(c) [15%] Linearity $\mathcal{L}(c_1 f + c_2 g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$ is one Laplace rule. State four other Laplace rules. Forward and backward table entries are not rules, which means $\mathcal{L}(1) = 1/s$ doesn't count.

(d) [25%] Solve by Laplace's resolvent method

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 2x(t), \end{aligned}$$

with initial conditions $x(0) = -1$, $y(0) = 2$.

(e) [25%] Derive $y(t) = \int_0^t \sin(t-u)f(u)du$ by Laplace transform methods from the forced oscillator problem

$$y''(t) + y(t) = f(t), \quad y(0) = y'(0) = 0.$$

Answer:

(a) Transform to obtain $\mathcal{L}(x) = \frac{s}{(s^2+1)^2}$.

Calculus method. Observe that $\frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$. Then $\mathcal{L}(x) = -\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+1} = -\frac{1}{2}\frac{d}{ds}\mathcal{L}(\sin t) = -\frac{1}{2}\mathcal{L}((-t)\sin t)$ by the s-differentiation theorem. Finally, $x(t) = \frac{1}{2}t\sin t$.

Convolution method. Write $\mathcal{L}(x) = \mathcal{L}(\sin t)\mathcal{L}(\cos t)$. Apply the convolution theorem to obtain $x(t) = \int_0^t \sin u \cos(t-u)du = \frac{1}{2}t\sin t$. A maple answer check is

```
int(sin(u)*cos(t-u),u=0..t);
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.

Hand integration uses the trigonometric identity $2\sin(a)\cos(b) = \cos(a-b) - \cos(a+b)$.

(b) No. The limit of the Laplace transform of a function of exponential order is zero as $s \to \infty$. The result $F(\infty) = 0$ is called the **Final Value Theorem**.

(c) The possible rules: Linearity, Lerch's cancelation law, parts formula, s-differentiation, first shift theorem, second shift theorem, periodic function formula, convolution theorem, delta function formula, integral theorem.

(d) The resolvent formula $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}_0$ becomes the $2\times$ system of equations

$$\begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Multiply by the inverse matrix of $(sI - A)$ on the left to obtain

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \end{pmatrix} = \begin{pmatrix} s-1 & -1 \\ -2 & s-0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} s-0 & 1 \\ 2 & s-1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

where $\Delta = \det(sI - A) = (s+1)(s-2)$. Then $\mathcal{L}(x) = \frac{2-s}{\Delta} = \frac{-1}{s+1}$, $\mathcal{L}(y) = \frac{2s}{\Delta} = \frac{2s-4}{(s+1)(s-2)} = \frac{2}{s+1}$. Then $x(t) = -e^{-t}$, $y(t) = 2e^{-t}$.

(e) Derive $y(t) = \int_0^t \sin(t-u)f(u)du$

Transform $y'' + y = f$ to get the transfer function relation

$$\mathcal{L}(y(t)) = \frac{1}{s^2+1}\mathcal{L}(f(t)) = \mathcal{L}(\sin t)\mathcal{L}(f(t)).$$

The convolution theorem implies the right side of the equation is $\mathcal{L}(\int_0^t \sin(t-u)f(u)du)$. Lerch's cancelation law implies $y(t) = \int_0^t \sin(t-u)f(u)du$.

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20. (ch7)

(a) [25%] Solve $\mathcal{L}(f(t)) = \dfrac{10}{(s^2+8)(s^2+4)}$ for $f(t)$.

(b) [25%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2(s+2)}$.

(c) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s-1}{s^2+2s+5}$.

(d) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(t^2 \sin 3t)$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^3 e^{9t}\cos 8t\right)\right)\Big|_{s\to s+3}.$$

Answer:

(a) $\mathcal{L}(f(t)) = \frac{10}{u+8}u+4$ where $u = s^2$. Use Heaviside's coverup method to find the partial fraction expansion

$$\frac{10}{u+8}u+4 = \frac{-5/2}{u+8} + \frac{5/2}{u+4} = \frac{-5/2}{s^2+8} + \frac{5/2}{s^2+4}.$$

Then $\mathcal{L}(f(t)) = \mathcal{L}\left(-\frac{5}{2}\frac{\sin\sqrt{8}t}{\sqrt{8}} + \frac{5}{2}\frac{\sin 2t}{2}\right)$ implies by Lerch's theorem

$$f(t) = -\frac{5}{2}\frac{\sin\sqrt{8}t}{\sqrt{8}} + \frac{5}{2}\frac{\sin 2t}{2}.$$

(b) Expand the fraction into partial fractions as follows:

$$\mathcal{L}(f) = \frac{s+1}{s^2(s+2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} = \mathcal{L}(a + bt + ce^{-2t}).$$

Then Lerch's theorem implies $f(t) = a + bt + ce^{-2t}$. The partial fraction constants are $a = 1/4, b = 1/2, c = -1/4$.

(d) Because $\frac{d}{ds}\mathcal{L}(g(t)) = \mathcal{L}((-t)g(t))$, then $\mathcal{L}(f) = \mathcal{L}((-t)t^2\sin 3t)$. Lerch's theorem implies $f(t) = -t^3\sin 3t$.

(e) The shifting theorem $\mathcal{L}(g(t))|_{s\to(s-a)} = \mathcal{L}(e^{at}g(t))$ is applied to remove the shift on the outside and put e^{-3t} into the Laplace integrand. Then $\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}t^3e^{9t}\cos 8t)$. Lerch's theorem implies $f(t) = t^3e^{6t}\cos 8t$.

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21. (ch9)

(a) Find the Fourier sine and cosine coefficients for the 2-periodic function $f(t)$ equal to $t/2$ on $0 \le t < 2$, $f(2) = 0$.

(b) State Fourier's convergence theorem.
(c) State the results for term-by-term integration and differentiation of Fourier series.

Answer:
(a) See the solution to problem 25 (a), *infra*.
(b) Use the statement in the textbook, Chapter 9. The requirement on f is $2T$-periodic and piecewise smooth. The formal Fourier series converges at every x to the average of the left and right hand limits of $f(x)$. See also Problem 27, *infra*.
(c) Use the statements in Chapter 9. Basically, integration always works, but differentiation fails except in expectional cases of extra smoothness of f.

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22. (ch9)
(a) Find a steady-state periodic solution using Fourier series and undetermined coefficients for $x'' + x = F(t)$, where $F(t)$ is 2-periodic and equal to 10 on $0 < t < 1$, equal to -10 on $1 < t < 2$.

(b) Display Fourier's Replacement Model for the solution to the heat problem $u_t = u_{xx}$, $u(0,t) = u(1,t) = 0$, $u(x,0) = f(x)$ on $0 \le x \le 1$, $t \ge 0$.

(c) Solve $u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = 80\sin^3 x$ on $0 \le x \le \pi$, $t \ge 0$.

Answer:
(a)
Because $F(t)$ equals 10 times the square wave on $-1 < t < 1$, then a standard Fourier series table can be used to write a Fourier series for $F(t)$. The method of undetermined coefficients supplies the answer for $x(t)$.
Some details. Only basic definitions and results for Fourier series will be used. Because $F(t)$ is odd and $L = 1$ in the standard Fourier coefficient formulas, then all cosine coefficients are zero and the sine coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} F(t)\sin\left(n\pi\frac{t}{L}\right)dt = \frac{1}{1}\left(2\int_0^1 10\sin(n\pi t)dt\right) = \frac{2(1-(-1)^n)}{n\pi}.$$

Undetermined coefficients assumes the output is

$$x(t) = \sum_{k=1}^{\infty} c_n \sin(n\pi t).$$

Substitution into the differential equation, using the rules for differentiation of Fourier series, gives the identity

$$\sum_{k=1}^{\infty} -k^2\pi^2\, c_k \sin(k\pi t) + \sum_{k=1}^{\infty} c_k \sin(k\pi t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

Matching coefficients of the sine terms left and right implies $(1 - k^2\pi^2)c_k = b_k$. Then

$$x(t) = \sum_{k=1}^{\infty} \frac{b_k}{1 - k^2\pi^2}\sin(k\pi t).$$

(b)
The eigenpairs for the problem are (λ_n, f_n), where $\lambda_n = (n\pi)^2$, $f_n(x) = \sin(\sqrt{\lambda_n} x)$. Write $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$. Then $u(x,t)$ is obtained from $f(x)$ by insertion of the exponential factor $e^{-\lambda_n t}$ after the sine factor:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-n^2\pi^2 t}.$$

(c)
The trick is to apply part (b), after writing $f(x)$ as a sine series. This uses trig identities, as follows.

$$\begin{aligned}\sin^3(x) &= (1-\cos^2(x))\sin(x) = \sin(x) - \tfrac{1}{2}\cos(x)(2\sin(x)\cos(x))\\ &= \sin(x) - \tfrac{1}{2}\cos(x)\sin(2x)\\ &= \sin(x) - \tfrac{1}{2}\tfrac{1}{2}\left(\sin(2x+x)+\sin(2x-x)\right)\\ &= \sin(x) - \tfrac{1}{4}\sin(3x) + \tfrac{1}{4}\sin(x)\\ &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x)\end{aligned}$$

Then

$$f(x) = 80\left(\frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\right) = 60\sin(x) - 20\sin(3x).$$

Fourier's replacement method then gives

$$u(x,t) = 60\sin(x)e^{-t} - 20\sin(3x)e^{-9t}.$$

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23. (Vibration of a Finite String)
The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes.

Solve the finite string vibration problem on $0 \le x \le 2$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx},\\ u(0,t) &= 0,\\ u(2,t) &= 0,\\ u(x,0) &= 0,\\ u_t(x,0) &= -11\sin(5\pi x).\end{aligned}$$

Answer: Because the wave initial shape is zero, then the only normal modes appearing

in the solution $u(x,t)$ are sine times sine.
The initial wave velocity is already a Fourier series, using orthogonal set $\{\sin(n\pi x/2)\}_{n=1}^{\infty}$. The 1-term Fourier series $-11\sin(5\pi x)$ can be modified into a solution by inserting an appropriate sine factor $\sin(5\pi t)$ present in the corresponding normal mode. The required initial velocity is $u_t(0,t) = -11\sin(5\pi x)$, so the sine factor has to be adjust by a constant k. We compute Because $\frac{d}{dt}\sin(5\pi t) = 5\pi\cos(5\pi t)$ at $t = 0$ is 5π, then $k = 1/(5\pi)$. Then $u(x,t) = -11\sin(5\pi x)\sin(5\pi ct)/(5\pi)$. We check it is a solution.

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24. (Periodic Functions)

(a) [30%] Find the period of $f(x) = \sin(x)\cos(2x) + \sin(2x)\cos(x)$.

(b) [40%] Let $p = 5$. If $f(x)$ is the odd $2p$-periodic extension to $(-\infty, \infty)$ of the function $f_0(x) = 100x\,e^{10x}$ on $0 \le x \le p$, then find $f(11.3)$. The answer is not to be simplified or evaluated to a decimal.

(c) [30%] Mark the expressions which are periodic with letter $\mathbf{P}$, those odd with $\mathbf{O}$ and those even with $\mathbf{E}$.

$$\sin(\cos(2x)) \qquad \ln|2+\sin(x)| \qquad \sin(2x)\cos(x) \qquad \frac{1+\sin(x)}{2+\cos(x)}$$

Answer: (a) $f(x) = \sin(x+2x)$ by a trig identity. Then period $= 2\pi/3$.
(b) $f(11.3) = f(11.3 - p - p) = f(1.3) = f_0(1.3) = 130e^{13}$.
(c) All are periodic of period 2π, satisfying $f(x+2\pi) = f(x)$. The first is even and the third is odd. The remaining functions are neither even nor odd.

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25. (Fourier Series)
Let $f_0(x) = x$ on the interval $0 < x < 2$, $f_0(x) = -x$ on $-2 < x < 0$, $f_0(x) = 0$ for $x = 0$, $f_0(x) = 2$ at $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [80%] Compute the Fourier coefficients of $f(x)$ (defined above) for the terms $\sin(67\pi x)$ and $\cos(2\pi x)$. Leave tedious integrations in integral form, but evaluate the easy ones like the integral of the square of sine or cosine.

(b) [20%] Which values of x in $|x| < 12$ might exhibit Gibb's over-shoot?

Answer: (a) Because $f_0(x)$ is even, then $f(x)$ is even. Then the coefficient of $\sin(67\pi x)$ is zero, without computation, because all sine terms in the Fourier series of f have zero coefficient. The coefficient of $\cos(n\pi x/2)$ for $n > 0$ is given by the formula

$$a_n = \frac{1}{2}\int_{-2}^{2} f_0(x)\cos(n\pi x/2)dx = \int_0^2 x\cos(n\pi x/2)dx.$$

For $\cos(2\pi x)$, we select $n\pi x/2 = 2\pi x$, or index $n = 4$.

(b) There are no jump discontinuities, f is continuous, so no Gibbs overshoot.

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27. (Convergence of Fourier Series)

(a) [40%] The Fourier Convergence Theorem for piecewise smooth functions applies to continuously differentiable functions of period $2p$. Re-state the Fourier Convergence Theorem for the special case of a $2p$-periodic continuously differentiable function. It is necessary to translate the results for interval $-\pi \le x \le \pi$ to the interval $-p \le x \le p$ and simplify the value to which the Fourier series converges.

(b) [30%] Give an example of a function $f(x)$ periodic of period 2 that has a Gibb's over-shoot at the integers $x = 0, \pm 2, \pm 4, \ldots$, (all $\pm 2n$) and nowhere else.

Answer: (a) Let f be a $2p$-periodic smooth function on $(-\infty, \infty)$. Then for all values of x,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/p) + b_n \sin(n\pi x/p),$$

where the Fourier coefficients a_0, a_n, b_n are given by the Euler formulas:

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx, \quad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(n\pi x/p)dx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(n\pi x/p)dx.$$

(b) Any 2-periodic continuous function f will work, if we alter the values of f at the desired points to produce a jump discontinuity. For example, define $f(x) = \sin(\pi x)$ except at the points $\pm 2n$, where $f(2n) = 2$ $(n = 0, 1, 2, 3, \ldots)$.
