Differential Equations 2280

Midterm Exam 3

Exam Date: 14 April 2017 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3

1. **(Linear Constant Equations of Order n)**

(a) [30%] Find by variation of parameters a particular solution y_p for the equation $y'' = x + x^2$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 256x = 247\cos(3t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time**, please don't convert your answer.

(c) [30%] Let $f(x) = x^2\cos(x) - x(e^x + 1)$. Find the characteristic equation of a linear homogeneous scalar differential equation of least order such that $y = f(x)$ is a solution.

Answers and Solution Details:

Problem 1: Chapter 3

Solution (a) Answer: $y_p = \dfrac{1}{6}x^3 + \dfrac{1}{12}x^4$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1y_2' - y_1'y_2 = 1$. Then for $f(x) = x + x^2$,

$y_p = y_1\displaystyle\int y_2\frac{-f}{W}dx + y_2\int y_1\frac{f}{W}dx$,

$y_p = 1\displaystyle\int -x(x + x^2)dx + x\int 1(x + x^2)dx$,

$y_p = -1\left(\dfrac{1}{3}x^3 + \dfrac{1}{4}x^4\right) + x\left(\dfrac{1}{2}x^2 + \dfrac{1}{3}x^3\right)$,

$y_p = \dfrac{1}{6}x^3 + \dfrac{1}{12}x^4$ Dropped terms for 1, x.

This answer is checked by quadrature, applied twice to $y'' = x + x^2$ with initial conditions zero.

Solution (b) The beats solution is identical to the superposition $x(t)$ of the undetermined coefficients solution $x_p(t) = d_1\cos(3t) + d_2\sin(3t)$ and the homogeneous solution $x_h(t) = c_1\cos(16t) + c_2\sin(16t)$, subject to $x(0) = x'(0) = 0$, which determines constants c_1, c_2. Find x_p first, then $d_1 = 1$, $d_2 = 0$. Use conditions $x(0) = x'(0) = 0$ to solve for $c_1 = -1$, $c_2 = 0$. The answer is Beats Solution $= x(t) = \cos(3t) - \cos(16t)$.

Solution (c) Write $f(x) = x^2\cos(x) - x(e^x + 1) = x^2\cos(x) - x - xe^x$, then identify Euler atoms $x, xe^x, x^2\cos(x)$. Euler's multiplicity theorem implies that the characteristic equation has to have roots $0, 0, 1, 1, i, i, i, -i, -i, -i$. For instance, solution xe^x implies $r = 1$ is a root of multiplicity 2. The root-factor theorem of college algebra implies that $(r-1)^2$ is a factor of the characteristic polynomial. The minimal characteristic polynomial is then, by repeated application of the root-factor theorem of college algebra, $r^2(r-1)^2(r-i)^3(r+i)^3$. The real form of the characteristic equation is $r^2(r-1)^2(r^2+1)^3 = 0$, which is a degree 10 polynomial equation.

Chapters 4 and 5

2. **(Systems of Differential Equations)**

(a) [30%] Assume a 3×3 matrix A has eigenvalues $\lambda = 3, 4, 5$. State the Cayley-Hamilton-Ziebur theorem for this example. Then display a solution formula for the vector solution $\vec{u}(t)$ to system $\frac{d}{dt}\vec{u} = A\vec{u}$, inserting what is known what is known from the eigenvalue information (supplied above).
(b) [40%] A linear cascade, typically found in brine tank models, satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 triangular matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Part 1. Use the linear integrating factor method to find the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Part 2. Laplace's method applies to this example. Explain in a paragraph of text how to apply Laplace's method to this 4×4 system. Don't use Laplace tables and don't find the solution! The explanation can use scalar equations or the vector-matrix equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Background for (c). Let A be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\vec{x}'(t) = A\vec{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\vec{x}(t) = \Phi(t)\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

(c) [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = x + 5y, \quad y' = -5x + y,$$

which has complex eigenvalues $\lambda = 1 \pm 5i$.

Part 1. Show the details of the method, finally displaying formulas for $x(t), y(t)$.
Part 2. Report a fundamental matrix $\Phi(t)$.
Part 3. Use **Part 2** to find the exponential matrix e^{At}.

Answers and Solution Details:
Problem 2: Chapters 4 and 5
Part (a) Cayley-Hamilton-Ziebur says that the solution of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ is a vector linear combination of the Euler solution atoms constructed from the roots of the characteristic equation $|A - \lambda I| = 0$. In this case, that means $\vec{x}(t) = \vec{c}_1 e^{2t} + \vec{c}_2 e^{2t} + \vec{c}_3 e^{2t}$.

Part (b)
(b) Part 1. The components x, y, z, w of the vector solution are given by formulas $x = c_2 t e^t + c_1 e^t$, $y = c_2 e^t$, $z = c_4 t e^{3t} + c_3 e^{3t}$, $w = c_4 e^{3t}$, details below.
Write the system in scalar form

$$\begin{aligned} x' &= x + y, \\ y' &= y, \\ z' &= 3z + w, \\ w' &= 3w. \end{aligned}$$

Solve the fourth equation $w' = 3w$ as
$w = \dfrac{\text{constant}}{\text{integrating factor}} = c_4 e^{3t}$.

$\boxed{w = c_4 e^{3t}}$
The third equation is
$z' = 3z + c_4 e^{3t}$
The linear integrating factor method applies.
$z' - 3z = c_4 e^{3t}$
$\dfrac{(Wz)'}{W} = c_4 e^{3t}$, where $W = e^{-3t}$,
$(Wz)' = c_4 W e^{3t}$
$(e^{-3t} z)' = c_4 e^{-3t} e^{3t}$
$e^{-3t} z = c_4 t + c_3$.
$\boxed{z = c_4 t e^{3t} + c_3 e^{3t}}$
Solve the second equation as
$y = \dfrac{\text{constant}}{\text{integrating factor}} = c_2 e^t$.
$\boxed{y = c_2 e^t}$
Stuff the expression into the first differential equation:
$x' = x + y = x + c_2 e^t$
Then solve with the linear integrating factor method.
$x' - x = c_2 e^t$
$\dfrac{(Wx)'}{W} = c_2 e^t$, where $W = e^{-t}$. Cross-multiply:
$(e^{-t} x)' = c_2 e^{-t} e^t$, then integrate:
$e^{-t} x = c_2\, t + c_1$
Then divide by e^{-t}:
$\boxed{x = c_2 t e^t + c_1 e^t}$

(b) Part 2.
The matrix of coefficients is not diagonalizable, therefore the eigenanalysis method fails to apply. However, Laplace's method applies anyway. The system is first transformed into $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}_0$ where $\vec{u}_0$ is a vector of arbitrary constants. Then $\mathcal{L}(\vec{u})$ is the inverse of $sI - A$ times $\vec{u}_0$. Inverse transforms are applied to the inverse matrix $(sI - A)^{-1}$, called the resolvent, to obtain $\vec{u} = \mathcal{L}^{-1}(sI - A)^{-1}\vec{u}_0$.

Solution (c) The equations $x' = x + 5y, \quad y' = -5x + y$ have coefficient matrix $A = \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix}$ with characteristic equation $(1 - \lambda)^2 + 25 = 0$. The roots are $1 \pm 5i$. The Euler atoms are $e^t \cos(5t), e^t \sin(5t)$.

(c) Part 1.
By C-H-Z, $x = c_1 e^t \cos(5t) + c_2 e^t \sin(5t)$. Isolate y from the first differential equation $x' = x + 5y$, obtaining the formula $5y = x' - x = x + e^t(-5c_1 \sin(5t) + 5c_2 \cos(5t)) - x = -5c_1 e^t \sin(5t) + 5c_2 e^t \cos(5t)$. Then the solution formulas are

$$x = c_1 e^t \cos(5t) + c_2 e^t \sin(5t), \quad y(t) = -c_1 e^t \sin(5t) + c_2 e^t \cos(5t).$$

(c) Part 2
A fundamental matrix $\Phi(t)$ is found by taking partial derivatives on the symbols c_1, c_2. The answer is exactly the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix}$ with respect to variables c_1, c_2.
$\Phi(t) = \begin{pmatrix} e^t \cos(5t) & e^t \sin(5t) \\ -e^t \sin(5t) & e^t \cos(5t) \end{pmatrix}$.

Chapter 6

3. (Linear and Nonlinear Dynamical Systems)

(a) [20%] Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \vec{u}$$

(b) [30%] Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - 2y + 32, \\ y' &= 2x(x - 2y). \end{aligned}$$

An equilibrium point is $x = -8$, $y = -4$. Compute the Jacobian matrix of the linearized system at this equilibrium point.

(c) [30%] Consider the soft nonlinear spring system $\begin{cases} x' &= y, \\ y' &= -5x - 2y + \frac{5}{4}x^3. \end{cases}$

> **(1)** Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$, where A is the Jacobian matrix of this system at $x = 2$, $y = 0$.
>
> **(2)** Apply the Pasting Theorem to classify $x = 2$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(d) [20%] State the hypotheses and the conclusions of the *Pasting Theorem* used in part (c) above. Accuracy and completeness expected.

Answers and Solution Details:

Problem 3: Chapter 6

Solution (a) Answer: **stable center**.

It is an unstable saddle. Details: The eigenvalues of A are roots of $r^2 + 1 = 0$, which are complex roots $i, -i$ with atoms $\cos t, \sin t$. Rotation eliminates both saddle and node. The atoms have no limit at either ∞ or $-\infty$, therefore it is a center.

Solution (b) The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -4y - 2 \\ 4x - 4y & -4x \end{pmatrix}$. Then $A = J(-8,-4) = \begin{pmatrix} 1 & 14 \\ -16 & 32 \end{pmatrix}$.

Solution (c)

(c) Part 1

The Jacobian is $J(x,y) = \begin{pmatrix} 0 & -2 \\ -5 + \frac{15}{4}x^2 & -2 \end{pmatrix}$. Then $A = J(0,0) = \begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$. The eigenvalues of A are found from $r^2 + 2r - 10 = 0$, giving real roots $-1 \pm \sqrt{11}$. Because no trig functions appear in the Euler solution atoms, then no rotation happens, and the classification must be a saddle or a nodel. The Euler solution atoms limit $(\infty, 0)$ $t = \infty$ and $(0, \infty)$ at $t = -\infty$, therefore the node is eliminated and it is a saddle. We report a **unstable saddle** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$.

(c) Part 2

Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system. Report: **unstable saddle** at $x = 0$, $y = 0$.

Solution (d)

The pasting theorem assumes that (x_0, y_0) is an isolated equilibrium point of the nonlinear system. The associated linear system is $\vec{u}' = A\vec{u}$, where A is the Jacobian matrix of the nonlinear system evaluated at this point. The theorem says that a spiral, center, saddle or node for the linear system corresponds to a spiral, center, saddle or node, respectively, with two exceptions.

Exception 1. If the equilibrium is a center for the linear problem, then the nonlinear system has either a center or a spiral at (x_0, y_0), and the spiral can be either stable or unstable.

Exception 2. If the equilibrium is a node for the linear problem with equal eigenvalues, then the nonlinear system has either a node or a spiral at (x_0, y_0). The stability of the linear problem transfers to the nonlinear problem.
