Differential Equations 2280

Midterm Exam 2 with Solutions

Exam Date: 31 March 2017 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Chapter 3)

(a) [70%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = 10\cos(t) - 100\sin(t)$.

(b) [30%] Linear algebra can find the solution of the current equation $I''+2I'+5I = 10\cos(t)-100\sin(t)$ having initial conditions $I(0) = 1$, $I'(0) = 0$. Write the linear algebraic equations for c_1, c_2, but to save time don't solve for c_1, c_2.

Answer:

Part (a) Answer: $I_{SS}(t) = 12\cos t - 19\sin t$.

Variation of Parameters.

To fit formulas for variation of parameters, we solve $x'' + 2x' + 5x = 10\cos(t) - 100\sin(t)$.
Solve $x'' + 2x' + 5x = 0$ to get $x_h = c_1x_1 + c_2x_2$, $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = 4e^{-2t}$. Then for $f(t) = 10\cos(t) - 100\sin(t)$,

$$x_p = x_1\int x_2\frac{-f}{W}dt + x_2\int x_1\frac{f}{W}dt.$$

The integrations are too difficult, so the method won't be pursued.

Undetermined Coefficients.

The trial solution by Rule I is $I = d_1\cos t + d_2\sin t$. The homogeneous solutions have exponential factors, therefore the Euler solution atoms in the trial solution cannot be solutions of the homogeneous problem, hence Rule II does not apply.
Substitute the trial solution into the non-homogeneous equation to obtain the answers $d_1 = 12$, $d_2 = -19$. The unique periodic solution I_{SS} is extracted from the general solution $I = I_h + I_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $I_p = d_1\cos t + d_2\sin t = 12\cos t - 19\sin t$ and the homogeneous solution x_h has negative exponential terms, then

$$I_{SS} = 12\cos t - 19\sin t.$$

Laplace Theory.

Plan: Find the general solution, then extract the steady-state solution by dropping negative exponential terms. The computation can use initial data $I(0) = I'(0) = 0$, because every particular solution contains the terms of the steady-state solution. Some details:

$(s^2 + 2s + 5)\mathcal{L}(I) = \dfrac{10s}{s^2+1} - \dfrac{100}{s^2+1}$

$\mathcal{L}(I) = \dfrac{10s - 100}{(s^2+1)(s^2+2s+5)}$

$\mathcal{L}(I) = \dfrac{10s - 100}{(s^2+1)((s+1)^2+4)}$

$\mathcal{L}(I) = \dfrac{as+b}{s^2+1} + \dfrac{c(s+1)+2d}{(s+1)^2+4}$

$\mathcal{L}(I) = a\mathcal{L}(\cos t) + b\mathcal{L}(\sin t) + c\mathcal{L}(e^{-t}\cos 2t) + d\mathcal{L}(e^{-t}\sin 2t)$

$I(t) = a\cos t + b\sin t + ce^{-t}\cos 2t + de^{-t}\sin 2t$, by Lerch's Theorem.

Dropping the negative exponential terms gives the steady-state solution $I_{SS}(t) = a\cos t + b\sin t$. Remaining is to find the constants a, b using partial fraction theory. The answers are $a = 12$, $b = -19$.

Part (b) Answer: $c_1 + 12 = 1$, $-c_1 + 2c_2 - 19 = 0$.

Linear Algebra.
The general solution is $I = I_h + I_p$ where $I_h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$ and $I_p(t) = 12\cos(t) - 19\sin(t)$ (from part (a)). The equations for c_1, c_2 obtained from $I(0) = 1, I'(0) = 0$ are

$$\begin{aligned} c_1 + 12 &= 1, \\ -c_1 + 2c_2 - 19 &= 0. \end{aligned}$$

It was not necessary to solve for c_1, c_2.

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2. (Laplace Theory)
(a) [40%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\left.\frac{d^2}{ds^2}\mathcal{L}(f(t))\right|_{s\to(s-3)} = \frac{1}{s^2} + \mathcal{L}(t^2 f(t) - t).$$

(b) [60%] Solve by Laplace's method $x'' + 2x' + x = e^{-t}$, $x(0) = x'(0) = 0$.

Answer:
(a)
Replace by the shift theorem and the s-differentiation theorem the given equation by

$$\mathcal{L}\left((-t)^2 f(t) e^{3t}\right) = \frac{1}{s^2} + \mathcal{L}(t^2 f(t) - t).$$

Replace $\frac{1}{s^2}$ by $\mathcal{L}(t)$, then collect terms on the right. Lerch's theorem cancels $\mathcal{L}$ to give $t^2 e^{3t} f(t) = t + t^2 f(t) - t$. Cancel terms to obtain

$$f(t) = 0.$$

(b)
The main steps are:
$(s^2 + 2s + 1)\mathcal{L}(y(t)) = \mathcal{L}\left(e^{-t}\right)$,
$\mathcal{L}(y(t)) = \dfrac{1}{(s+1)^2}\mathcal{L}\left(e^{-t}\right)$,
$\mathcal{L}(y(t)) = \dfrac{1}{(s+1)^3}$,
$\mathcal{L}(y(t)) = \left.\dfrac{1}{s^3}\right|_{s:=s+1}$, a shift substitution,
$\mathcal{L}(y(t)) = \left.\mathcal{L}\left(\dfrac{t^2}{2}\right)\right|_{s:=s+1}$, by the backward Laplace table,
$\mathcal{L}(y(t)) = \mathcal{L}\left(\dfrac{t^2}{2}e^{-t}\right)$, by the shifting theorem,
$y(t) = \dfrac{1}{2}t^2 e^{-t}$, by Lerch's theorem.

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3. (Laplace Theory)
(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{10/s}{(s^2+1)(s^2+5)}$ for $f(t)$.

(b) [30%] Solve $x''' + x' = 0$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = 0$ by Laplace's Method.

(c) [40%] Solve the system $x' = 4x + y + 30$, $y' = x + 4y + 60$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:

(3a) $\mathcal{L}(f) = \dfrac{100}{s(s^2+1)(s^2+4)} = \dfrac{a}{s} + \dfrac{bs+c}{s^2+1} + \dfrac{ds+e}{s^2+4}$. Then $\mathcal{L}(f) = \mathcal{L}\left(a + b\cos(t) + c\sin t + d\cos(2t) + \frac{1}{2}e\sin(2t\right.$ implies by Lerch's theorem $f(t) = a + b\cos(t) + c\sin t + d\cos(2t) + \frac{1}{2}e\sin(2t)$. It remains to find by partial fractions the answers for the constants: $a = 25/4, b = -100/3, c = 0, d = 25/3, e = 0$.

(3b) The answer is $x(t) = 1 + \sin(t)$.

The transformed equation is $(s^3 + s)\mathcal{L}(x) = 1 + s + s^2$. Then $\mathcal{L}(x) = \dfrac{1+s+s^2}{s(s^2+1)} = \dfrac{a}{s} + \dfrac{bs+c}{s^2+1} = \mathcal{L}\left(a + b\cos(t) + c\sin(t)\right)$. Lerch's theorem implies $x(t) = a + b\cos(t) + c\sin(t)$. Partial fractions methods apply to solve for $a = 1, b = 0, c = 1$. Then $x(t) = 1 + \sin(t)$.

(3c) The answer is $x(t) = -5e^{3t} + 9e^{5t} - 4$, $y(t) = 5e^{3t} + 9e^{5t} - 14$.

The transformed system $\vec{u}' = A\vec{u} + \vec{b}$, with $A = \begin{pmatrix} 4 & 1 \\ 5 & 9 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 30 \\ 60 \end{pmatrix}$, is given by

$$\begin{array}{rcrcl} (s-4)\mathcal{L}(x) & + & (-1)\mathcal{L}(y) & = & \mathcal{L}(30), \\ (-1)\mathcal{L}(x) & + & (s-4)\mathcal{L}(y) & = & \mathcal{L}(60). \end{array}$$

Then $\mathcal{L}(1) = 1/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{30\,s - 60}{(s-3)\,(s-5)\,s}, \quad \mathcal{L}(y) = \frac{-210 + 60\,s}{(s-3)\,(s-5)\,s}.$$

After partial fractions and the backward table,

$$x(t) = -5e^{3t} + 9e^{5t} - 4, \quad y(t) = 5e^{3t} + 9e^{5t} - 14.$$

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4. (Systems of Differential Equations)

The Eigenanalysis Method (section 5.2) says that, for a 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$, the general solution is $\vec{u}(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} + c_3\vec{v}_3e^{\lambda_3 t}$. In the solution formula, $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, $(\lambda_3, \vec{v}_3)$ are eigenpairs of A. Assume given the 3×3 matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) [50%] Matrix A has only two eigenpairs. Display eigenanalysis details for A.

(b) [25%] It is impossible to apply the Eigenanalysis Method (stated above). Explain why.

(c) [25%] Display the solution of $\frac{d}{dt}\vec{u} = A\vec{u}$ in case A is 4×4 and has eigenvalues $2, -1, 3, 5$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Answer:

(a): The details should solve the equation $|A - \lambda I| = 0$ for three eigenvalues $\lambda = 5, 4, 4$. Then solve the systems $(A - (5)I)\vec{v} = \vec{0}$ for eigenvector $\vec{v} = \vec{v}_1$ and $(A - (4)I)\vec{v} = \vec{0}$ for eigenvector $\vec{v} = \vec{v}_2$. There is no

eigenvector $\vec{v}_3$.
The eigenpairs are

$$\left(5, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

(b): The eigenanalysis method applies only in case the matrix is diagonalizable. But A has only 2 eigenpairs, so it is not diagonalizable.
(c): The eigenanalysis method implies

$$\vec{u}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_4 e^{5t} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

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5. (Systems of Differential Equations)
Systems $\frac{d}{dt}\vec{u} = A\vec{u}$ with A an $n \times n$ real matrix can be solved by the following methods:

(1) Cayley-Hamilton-Ziebur method, from section 4.2. (2) Eigenanalysis method from 5.2. (3) Laplace's method, from chapter 7. (4) Exponential matrix, from 5.6

(a) [50%] The eigenvalues are 4, 6 for the matrix $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$. Display the general solution of $\frac{d}{dt}\vec{u} = A\vec{u}$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5).

(b) [10%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is supplied with matrix A having only two eigenpairs. It can be solved using the exponential matrix. What other methods are possible to use? Don't do any details, write a sentence.

(c) [10%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is supplied with matrix A having three eigenpairs, but only one real eigenvalue. It can be solved using the exponential matrix. What other methods are possible to use? Don't do any details, write a sentence.

(d) [30%] The 3×3 system $\frac{d}{dt}\vec{u} = A\vec{u}$ is given by $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Choose a method other than the exponential matrix and explain how you would solve for $\vec{u}$. It is not necessary to find the answer, but it is necessary to outline the method, not omitting any details.

Answer:
(a) **Cayley-Hamilton Ziebur Shortcut**. The method says that the components $x(t), y(t)$ of the solution to the system

$$\frac{d}{dt}\vec{u} = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ are linear combinations of the Euler atoms found from the roots of the characteristic equation $|A - rI| = 0$. The roots are $r = 4, 6$ and the atoms are e^{4t}, e^{6t}. The scalar system is

$$\begin{cases} x'(t) &=& 5x(t) + y(t), \\ y'(t) &=& x(t) + 5y(t), \\ x(0) &=& 1, \\ y(0) &=& -1. \end{cases}$$

The C-H-Z method implies $x(t) = c_1e^{4t} + c_2e^{6t}$, but c_1, c_2 are not arbitrary constants: they are determined by the initial conditions $x(0) = 1, y(0) = -1$. Then $x' = 5x+y$ can be solved for y to obtain $y(t) = x'(t) - 5x(t)$. Substitute expression $x(t) = c_1e^{4t} + c_2e^{6t}$ into $y(t) = x'(t) - 5x(t)$ to obtain

$$y(t) = x'(t) - 5x(t) = 4c_1e^{4t} + 6c_2e^{6t} - 5(c_1e^{4t} + c_2e^{6t}) = -c_1e^{4t} + c_2e^{6t}.$$

Then

$$(1) \qquad \begin{cases} x(t) &= \quad c_1e^{4t} &+ \quad c_2e^{6t}, \\ y(t) &= -c_1e^{4t} &+ \quad c_2e^{6t}. \end{cases}$$

Initial data $x(0) = 1, y(0) = -1$ are used in the last step, to evaluate c_1, c_2. Inserting these conditions produces a 2×2 linear system for c_1, c_2

$$\begin{cases} 1 &= \quad c_1e^0 &+ \quad c_2e^0, \\ -1 &= -c_1e^0 &+ \quad c_2e^0. \end{cases}$$

The solution is $c_1 = 1$ and $c_2 = 0$, which implies the final answer $x(t) = e^{4t}$, $y(t) = -e^{6t}$.
(b) Available are methods (1) and (3). The eigenanalysis method does not apply, because the matrix is not diagonalizable. Other methods are possible, for example the linear integrating factor method for cascades, provided the matrix is triangular.
(c) Methods (1), (2) and (3) apply.
(d) Any of methods (1), (2), (3) apply because the matrix is diagonalizable. The triangular matrix has eigenvalues $1, 1, 0$ and eigenpairs

$$\left(1, \left(\begin{array}{c} 1, 0, 0 \end{array} \right)\right), \left(1, \left(\begin{array}{c} 0, 1, 0 \end{array} \right)\right), \left(0, \left(\begin{array}{c} -1, -1, 1 \end{array} \right)\right).$$

By the eigenanalysis method,

$$\vec{u} = c_1e^t \left(\begin{array}{c} 1, 0, 0 \end{array} \right) + c_2e^t \left(\begin{array}{c} 0, 1, 0 \end{array} \right) + e^0 \left(\begin{array}{c} -1, -1, 1 \end{array} \right).$$

Remark on Fundamental Matrices. The answer prior to evaluation of c_1, c_2 can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The matrix $\Phi(t) = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix}$ is called a **fundamental matrix**, because it is nonsingular and satisfies $\Phi' = A\Phi$ (its columns are solutions of $\frac{d}{dt}\vec{u} = A\vec{u}$). In terms of Φ,

$$e^{At} = \Phi(t)\Phi^{-1}(0).$$

This formula gives an alternative way to compute e^{At} by using the Cayley-Hamilton-Ziebur shortcut. Please observe that the columns of Φ are the formal partial derivatives of the vector solution $\vec{u}$ on the symbols c_1, c_2. Partial derivatives on symbols is a general method for discovering basis vectors. Therefore, Φ can be written directly from equations (1).