Differential Equations 2280

Sample Midterm Exam 2 with Solutions
Exam Date: 5 April 2019 at 7:30am

Instructions: This in-class exam is 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exam 1. Exam 2 covers only problems 1-7, which is Chapters 1 to 5 and 7 in the textbook. Chapter 6 (Problem 8) is moved to the final exam.

1. **(Laplace Theory)**

(a) [50%] Solve by Laplace's method $x'' + 2x' + x = e^t$, $x(0) = x'(0) = 0$.

(b) [25%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\frac{d}{ds}\mathcal{L}(f(t))\Big|_{s\to(s-3)} = \mathcal{L}(f(t) - t).$$

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods, explicitly using the Convolution Theorem, for the problem

$$y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0.$$

This is similar to a required homework problem from Chapter 7.

Answer:

(a)

$x(t) = -1/4\,e^{-t} - 1/2\,e^{-t}t + 1/4\,e^t$

An intermediate step is $\mathcal{L}(x(t)) = \dfrac{1}{(s-1)(s+1)^2}$. The solution uses partial fractions $\dfrac{1}{(s-1)(s+1)^2} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$, with answers $A = 1/4$, $B = -1/4$, $C = -1/2$.

(b)

Replace by the shift theorem and the s-differentiation theorem the given equation by

$$\mathcal{L}\left((-t)f(t)e^{3t}\right) = \mathcal{L}(f(t) - t).$$

Then Lerch's theorem cancels $\mathcal{L}$ to give $-te^{3t}f(t) = f(t) - t$. Solve for

$$f(t) = \frac{t}{1 + te^{3t}}.$$

(c)

The main steps are:

$(s^2 + 4s + 4)\mathcal{L}(y(t)) = \mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \dfrac{1}{(s+2)^2}\mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \mathcal{L}(te^{-2t})\mathcal{L}(f(t))$, by the first shifting theorem,

$\mathcal{L}(y(t)) = \mathcal{L}(\text{convolution of } te^{-2t} \text{ and } f(t))$, by the Convolution Theorem,

$\mathcal{L}(y(t)) = \mathcal{L}\left(\int_0^t xe^{-2x}f(t-x)dx\right)$, insert definition of convolution,

$y(t) = \displaystyle\int_0^t xe^{-2x}f(t-x)dx$, by Lerch's Theorem.

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2. **(Laplace Theory)**

(4a) [20%] Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$. Reference only. Not to appear on any exam.

(4b) [15%] Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$.

(4c) [15%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)}$.

(4d) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(4e) [20%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ by Laplace's Method.

(4f) [20%] Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:

(4a) Laplace's method explained.

The first step transforms the equation using the parts formula and initial data to get

$$(s+2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to evaluate $\mathcal{L}(e^t)$. Then write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s-1)}{s+2} = \frac{s}{(s-1)(s+2)}.$$

Partial fraction methods plus the backward Laplace table imply

$$\mathcal{L}(x) = \frac{a}{s-1} + \frac{b}{s+2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

(4b) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4}) = \frac{100}{3}\mathcal{L}(\sin t - \frac{1}{2}\sin 2t)$ implies $f(t) = \frac{100}{3}(\sin t - \frac{1}{2}\sin 2t)$.

(4c) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+3} = \mathcal{L}(a + bt + ce^{-3t})$ implies $f(t) = a + bt + ce^{-3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = 1/3$, $c = 1/9$.

(4d) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \dfrac{d}{ds}\left(\dfrac{1}{(s-2)^2+9}\right) = \dfrac{-2(s-2)}{((s-2)^2+9)^2}$.

(4e) The answer is $x(t) = 1$, by guessing, then checking the answer. The Laplace details jump through hoops to arrive at $(s^3 + s^2)\mathcal{L}(x(t)) = s^2 + s$, or simply $\mathcal{L}(x(t)) = 1/s$. Then $x(t) = 1$.

(4f) The transformed system is

$$\begin{aligned} (s-1)\mathcal{L}(x) &+ (-1)\mathcal{L}(y) &= 0, \\ (-1)\mathcal{L}(x) &+ (s+1)\mathcal{L}(y) &= \mathcal{L}(2). \end{aligned}$$

Then $\mathcal{L}(2) = 2/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{2}{s(s^2-2)}, \quad \mathcal{L}(y) = \frac{2(s-1)}{s(s^2-2)}.$$

After partial fractions and the backward table,

$$x = -1 + \cosh(\sqrt{2}t), \quad y = \sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t) + 1.$$

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3. **(Laplace Theory)**

(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+s)(s^2-s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+5}$.

(c) [20%] Let $u(t)$ denote the unit step. Solve for $f(t)$ in the relation

$$\mathcal{L}(f(t)) = \frac{d}{ds}\mathcal{L}(u(t-1)\sin 2t)$$

Remark: This is not a second shifting theorem problem.

(d) [30%] Compute $\mathcal{L}(e^{2t}f(t))$ for

$$f(t) = \frac{e^t - e^{-t}}{t}.$$

Answer:

(a) $f(t) = \sinh(t) - t = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - t$

(b) $f(t) = e^{-2t}(\cos(t) - \sin(t))$

(c) Replace d/ds by factor $(-t)$ in the Laplace integrand:

$$\mathcal{L}(f(t)) = \mathcal{L}((-t)\sin(2t)u(t-1))$$

Apply Lerch's theorem to cancel $\mathcal{L}$ on each side, obtaining the answer

$$f(t)) = (-t)\sin(2t)u(t-1).$$

(d) The first shifting theorem reduces the problem to computing $\mathcal{L}(f(t))$.

$\mathcal{L}(tf(t)) = \mathcal{L}(e^t - e^{-t}) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$

$-\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$, by the s-differentiation theorem,

Then $F(s) = \mathcal{L}(f(t))$ satisfies a first order quadrature equation $F'(s) = h(s)$ with solution $F(s) = \ln|s+1| - \ln|s-1| + c = \ln\left|\frac{s+1}{s-1}\right| + c$ for some constant c. Because $F = 0$ at $s = \infty$ (a basic theorem for functions of exponential order) and $\ln|1| = 0$, then $c = 0$ and $\mathcal{L}(f(t)) = F(s) = \ln|s+1| - \ln|s-1| = \ln\left|\frac{s+1}{s-1}\right|$. Then the shifting theorem implies

$$\mathcal{L}\left(e^{2t}f(t)\right) = \mathcal{L}(f(t))|_{s:=s-2} = \ln\left|\frac{s-1}{s-3}\right|.$$

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4. **(Systems of Differential Equations)**

The eigenanalysis method says that, for a 3×3 system $\mathbf{x}' = A\mathbf{x}$, the general solution is $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for A.

(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) Repeat (a), (b) for the matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix}$.

Answer:

(a): The details should solve the equation $|A - \lambda I| = 0$ for three values $\lambda = 5, 4, 3$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 5, 4, 3$.

The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(b): The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(c): The eigenpairs are

$$6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 4, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

and the eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

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5. **(Systems of Differential Equations)**

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

(b) [20%] Justify that eigenvectors of A corresponding to the eigenvalues in increasing order are the four vectors

$$\begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) [50%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis method.

Answer:

(a) Eigenvalues are $\lambda = 2, 3, 4, 5$.

Define

$$A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ to obtain the characteristic polynomial $(2 - \lambda)(3 - \lambda)(4 - \lambda)(5 - \lambda) = 0$. The eigenvalues are the roots: $\lambda = 2, 3, 4, 5$.

Used here was the *cofactor rule* for determinants. Also possible is the special result for block matrices, $\begin{vmatrix} B_1 & 0 \\ C & B_2 \end{vmatrix} = |B_1||B_2|$. Sarrus' rule does not apply for 4×4 determinants (an error) and the triangular rule likewise does not directly apply (another error).

(b) To be justified is $AP = PD$ where $D = \mathbf{diag}(2, 3, 4, 5)$ is the diagonal matrix of eigenvalues (see part (a)) and P is the augmented matrix of eigenvectors supplied above. Matrix multiply can check the answer, by expanding each side of $AP = PD$.

Alternative method:

Solve $(A - \lambda I)\vec{v} = \vec{0}$ four times, for $\lambda = 2, 3, 4, 5$. Each is a homogeneous system of linear algebraic equations, reduced to RREF by swap, combo, multiply. Each eigenvector answer is Strang's Special Solution.

(c) Because the eigenvalues are $\lambda = 2, 3, 4, 5$, then the solution is a vector linear combination of the Euler solution atoms $e^{2t}, e^{3t}, e^{4t}, e^{5t}$:

$$\mathbf{u}(t) = \vec{d}_1 e^{2t} + \vec{d}_2 e^{3t} + \vec{d}_3 e^{4t} + \vec{d}_4 e^{5t}.$$

According to the theory, $\vec{d}_j = c_j \vec{v}_j$, where $(\lambda_1, \vec{v}_1)$, ..., $(\lambda_4, \vec{v}_4)$ are the eigenpairs of A and c_1, c_2, c_3, c_4 are invented symbols representing real, arbitrary constants. Then

$$\vec{u} = c_1 e^{2t} \begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

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6. (Systems of Differential Equations)

(a) [100%] The eigenvalues are 3, 5 for the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5). Assume initial condition $\vec{u}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Answer:

(a) **Cayley-Hamilton Ziebur Shortcut**. The method says that the components $x(t), y(t)$ of the solution to the system

$$\vec{u}' = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ are linear combinations of the Euler atoms found from the roots of the characteristic equation $|A - rI| = 0$. The roots are $r = 3, 5$ and the atoms are e^{3t}, e^{5t}. The scalar system is

$$\begin{cases} x'(t) &= 4x(t) + y(t), \\ y'(t) &= x(t) + 4y(t), \\ x(0) &= 1, \\ y(0) &= -1. \end{cases}$$

The C-H-Z method implies $x(t) = c_1 e^{3t} + c_2 e^{5t}$, but c_1, c_2 are not arbitrary constants: they are determined by the initial conditions $x(0) = 1, y(0) = -1$. Then $x' = 4x + y$ can be solved for y to obtain $y(t) = x'(t) - 4x(t)$. Substitute expression $x(t) = c_1 e^{3t} + c_2 e^{5t}$ into $y(t) = x'(t) - 4x(t)$ to obtain

$$y(t) = x'(t) - 4x(t) = 3c_1 e^{3t} + 5c_2 e^{5t} - 4(c_1 e^{3t} + c_2 e^{5t}) = -c_1 e^{3t} + c_2 e^{5t}.$$

Then

(1)
$$\begin{cases} x(t) &= c_1 e^{3t} + c_2 e^{5t}, \\ y(t) &= -c_1 e^{3t} + c_2 e^{5t}. \end{cases}$$

Initial data $x(0) = 1, y(0) = -1$ are used in the last step, to evaluate c_1, c_2. Inserting these conditions produces a 2×2 linear system for c_1, c_2

$$\begin{cases} 1 &= c_1 e^0 + c_2 e^0, \\ -1 &= -c_1 e^0 + c_2 e^0. \end{cases}$$

The solution is $c_1 = 1$ and $c_2 = 0$, which implies the final answer $x(t) = e^{3t}$, $y(t) = -e^{3t}$.

Remark on Fundamental Matrices. The answer prior to evaluation of c_1, c_2 can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The matrix $\Phi(t) = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix}$ is called a **fundamental matrix**, because it is nonsingular and satisfies $\Phi' = A\Phi$ (its columns are solutions of $\vec{u}' = A\vec{u}$). In terms of Φ,

$$e^{At} = \Phi(t)\Phi^{-1}(0).$$

This formula gives an alternative way to compute e^{At} by using the Cayley-Hamilton-Ziebur shortcut. Please observe that the columns of Φ are the formal partial derivatives of the vector solution $\vec{u}$ on the symbols c_1, c_2. Partial derivatives on symbols is a general method for discovering basis vectors. Therefore, Φ can be written directly from equations (1).

Chapters 4 and 5

7. (Systems of Differential Equations)

Background. Let A be a real 3×3 matrix with eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $(\lambda_3, \mathbf{v}_3)$. The eigenanalysis method says that the 3×3 system $\mathbf{x}' = A\mathbf{x}$ has general solution

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t}.$$

Background. Let A be an $n \times n$ real matrix. The method called **Cayley-Hamilton-Ziebur** is based upon the result

> The components of solution $\mathbf{x}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Background. Let A be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

(a) Display eigenanalysis details for the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) The 3×3 triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models. Using the linear integrating factor method, starting with component $x_3(t)$, find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) The exponential matrix e^{At} is defined to be a fundamental matrix $\Psi(t)$ selected such that $\Psi(0) = I$, the $n \times n$ identity matrix. Justify the formula $e^{At} = \Phi(t)\Phi(0)^{-1}$, valid for *any* fundamental matrix $\Phi(t)$.

(d) Let A denote a 2×2 matrix. Assume $\mathbf{x}'(t) = A\mathbf{x}(t)$ has scalar general solution $x_1 = c_1 e^t + c_2 e^{2t}$, $x_2 = (c_1 - c_2)e^t + 2c_1 + c_2)e^{2t}$, where c_1, c_2 are arbitrary constants. Find a fundamental matrix $\Phi(t)$ and then go on to find e^{At} from the formula in part (c) above.

(e) Let A denote a 2×2 matrix and consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Assume fundamental matrix $\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ 2e^t & -e^{2t} \end{pmatrix}$. Find the 2×2 matrix A.

(f) The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 3x + y, \quad y' = -x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm i$. Show the details of the method, then go on to report a fundamental matrix $\Phi(t)$.

Remark. The vector general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, which contains no complex numbers. Reference: 4.1, Examples 6,7,8.

Answer:

Part (a) The details should solve the equation $|A - \lambda I| = 0$ for the three eigenvalues $\lambda = 5, 4, 3$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 5, 4, 3$.
The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Part (b) Write the system in scalar form

$$\begin{aligned} x' &= 3x + y + z, \\ y' &= 4y + z, \\ z' &= 5z. \end{aligned}$$

Solve the last equation as
$z = \dfrac{\text{constant}}{\text{integrating factor}} = c_3 e^{5t}$.
$\boxed{z = c_3 e^{5t}}$
The second equation is
$y' = 4y + c_3 e^{5t}$
The linear integrating factor method applies.
$y' - 4y = c_3 e^{-5t}$
$\dfrac{(Wy)'}{W} = c_3 e^{5t}$, where $W = e^{-4t}$,
$(Wy)' = c_3 W e^{5t}$
$(e^{-4t} y)' = c_3 e^{-4t} e^{5t}$
$e^{-4t} y = c_3 e^{t} + c_2$.
$\boxed{y = c_3 e^{5t} + c_2 e^{4t}}$
Stuff these two expressions into the first differential equation:
$x' = 3x + y + z = 3x + 2c_3 e^{5t} + c_2 e^{4t}$
Then solve with the linear integrating factor method.
$x' - 3x = 2c_3 e^{5t} + c_2 e^{4t}$
$\dfrac{(Wx)'}{W} = 2c_3 e^{5t} + c_2 e^{4t}$, where $W = e^{-3t}$. Cross-multiply:
$(e^{-3t} x)' = 2c_3 e^{5t} e^{-3t} + c_2 e^{4t} e^{-3t}$, then integrate:
$e^{-3t} x = c_3 e^{2t} + c_2 e^{t} + c_1$
$e^{-3t} x = c_3 e^{2t} + c_2 e^{t} + c_1$, divide by e^{-3t}:
$\boxed{x = c_3 e^{5t} + c_2 e^{4t} + c_1 e^{3t}}$

Part (c) The question reduces to showing that e^{At} and $\Phi(t)\Phi(0)^{-1}$ have equal columns. This is done by showing that the matching columns are solutions of $\vec{u}\,' = A\vec{u}$ with the same initial condition $\vec{u}(0)$, then apply Picard's theorem on uniqueness of initial value problems.

Part (d) Take partial derivatives on the symbols c_1, c_2 to find vector solutions $\vec{v}_1(t)$, $\vec{v}_2(t)$. Define $\Phi(t)$ to be the augmented matrix of $\vec{v}_1(t)$, $\vec{v}_2(t)$. Compute $\Phi(0)^{-1}$, then multiply on the right of $\Phi(t)$ to obtain $e^{At} = \Phi(t)\Phi(0)^{-1}$. Check the answer in a computer algebra system or using Putzer's formula.

Part (e) The equation $\Phi'(t) = A\Phi(t)$ holds for every t. Choose $t = 0$ and then solve for $A = \Phi'(0)\Phi(0)^{-1}$.

Part (f) By C-H-Z, $x = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$. Isolate y from the first differential equation $x' = 3x + y$, obtaining the formula $y = x' - 3x = -c_1 e^{3t}\sin(t) + c_2 e^{3t}\cos(t)$. A fundamental matrix is found by taking partial derivatives on the symbols c_1, c_2. The answer is exactly the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix}$ with respect to variables c_1, c_2.

$\Phi(t) = \begin{pmatrix} e^{3t}\cos(t) & e^{3t}\sin(t) \\ -e^{3t}\sin(t) & e^{3t}\cos(t) \end{pmatrix}$.

Chapter 6

8. **(Linear and Nonlinear Dynamical Systems)**

(a) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

(b) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}$$

(c) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4, 4)$ of the linearized system at this equilibrium point.

(d) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -x - 2y^2 - y + 32, \\ y' &= 2x^2 + 2xy. \end{aligned}$$

An equilibrium point is $x = -4$, $y = 4$. Compute the Jacobian matrix $A = J(-4, 4)$ of the linearized system at this equilibrium point.

(e) Consider the nonlinear dynamical system $\begin{cases} x' = -4x + 4y + 9 - x^2, \\ y' = 3x - 3y. \end{cases}$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply the Pasting Theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(f) Consider the nonlinear dynamical system $\begin{cases} x' &= -4x - 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{cases}$

At equilibrium point $x = 3$, $y = -3$, the Jacobian matrix is $A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$.

Linearization. Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$.

Nonlinear System. Apply the Pasting Theorem to classify $x = 3$, $y = -3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Answer:

Part (a) It is an unstable spiral. Details: The eigenvalues of A are roots of $r^2 - 2r + 5 = (r-1)^2 + 4 = 0$, which are complex conjugate roots $1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^t \cos 2t$, $e^t \sin 2t$ have limit zero at $t = -\infty$, therefore the system is stable at $t = -\infty$ and unstable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

Part (b) It is a stable spiral. Details: The eigenvalues of A are roots of $r^2 + 2r + 5 = (r+1)^2 + 4 = 0$, which are complex conjugate roots $-1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t} \cos 2t$, $e^{-t} \sin 2t$ have limit zero at $t = \infty$, therefore the system is stable at $t = \infty$ and unstable at $t = -\infty$. So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

Part (c) The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -4y-1 \\ 4x-2y & -2x \end{pmatrix}$. Then $A = J(4,4) = \begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$.

Part (d) The Jacobian is $J(x,y) = \begin{pmatrix} -1 & -4y-1 \\ 4x+2y & 2x \end{pmatrix}$. Then $A = J(-4,4) = \begin{pmatrix} -1 & -17 \\ -8 & -8 \end{pmatrix}$.

Part (e) (1) The Jacobian is $J(x,y) = \begin{pmatrix} -4-2x & 4 \\ 3 & -3 \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 + 13r + 18 = 0$, giving distinct real negative roots $-\frac{13}{2} \pm (\frac{1}{2})\sqrt{97}$. Because there are no trig functions in the Euler solution aistoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at $t = \infty$, therefore it is a node and we report a **stable node** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$.
(2) Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **stable node** at $x = 3$, $y = 3$. The exceptional case in Theorem 2 is a proper node, having characteristic equation roots that are equal. Stability is always preserved for nodes.

Part (f)
Linearization. The Jacobian is $J(x,y) = \begin{pmatrix} -4-2x & -4 \\ 3 & 3 \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$. The eigenvalues of A are found from $r^2 + 7r - 18 = 0$, giving distinct real roots $2, -9$. Because there are no trig functions in the Euler solution atoms e^{2t}, e^{-9t}, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms do not limit to zero at $t = \infty$ or $t = -\infty$, therefore it is a saddle and we report a **unstable saddle** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$.
Nonlinear System. Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system: **unstable saddle** at $x = 3$, $y = 3-$.

Final Exam Problems

Chapter 3: Linear Constant Equations of Order n.

(a) Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) A particular solution of the equation $mx'' + cx' + kx = F_0 \cos(2t)$ happens to be $x(t) = 11 \cos 2t + e^{-t} \sin \sqrt{11}t - \sqrt{11} \sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution $x_{\rm SS}$.

(c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and xe^{3x}. Write a formula for the general solution.

(d) Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time, don't convert to phase-amplitude form**.

(e) Write the solution $x(t)$ of

$$x''(t) + 25x(t) = 180\sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
To save time, don't convert to phase-amplitude form.

(f) Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 2x = 5\sin(t)$.

(g) Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

(h) Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

(i) Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

(j) Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Chapter 5. Solve a homogeneous system $u' = Au$, $u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ using the matrix exponential, Zeibur's method, Laplace resolvent and eigenanalysis.

Chapter 5. Solve a non-homogeneous system $u' = Au + F(t)$, $u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $F(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ using variation of parameters.

Answer: Chapter 3 final exam sample solutions.

Part (a) Answer: $y_p = \frac{x^2}{2} - \frac{x^3}{6}$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1y_2' - y_1'y_2 = 1$. Then for $f(t) = 1 - x$,

$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx,$

$y_p = 1 \int -x(1-x)dx + x \int 1(1-x)dx,$

$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2),$

$y_p = x^2/2 - x^3/6.$

This answer is checked by quadrature, applied twice to $y'' = 1 - x$ with initial conditions zero.

Part (b) It has to be the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then $x_{\text{SS}}(t) = 11\cos 2t - \sqrt{11}\sin 2t$.

Part (c) In order for xe^{3x} to be a solution, the general solution must have Euler atoms e^{3x}, xe^{3x}. Then the first solution $2e^{3x} + 4x$ minus 2 times the Euler atom e^{3x} must be a solution, therefore x is a solution, resulting in Euler atoms $1, x$. The general solution is then a linear combination of the four Euler atoms: $y = c_1(1) + c_2(x) + c_3(e^{3x}) + c_4(xe^{3x})$.

Part (d) Use undetermined coefficients trial solution $x = d_1\cos 4t + d_2\sin 4t$. Then $d_1 = 5/6$, $d_2 = 0$, and finally $x_p(t) = (5/6)\cos(4t)$. The characteristic equation $r^2 + 64 = 0$ has roots $\pm 8i$ with corresponding Euler solution atoms $\cos(8t), \sin(8t)$. Then $x_h(t) = c_1\cos(8t) + c_2\sin(8t)$. The general solution is $x = x_h + x_p$. Now use $x(0) = x'(0) = 0$ to determine $c_1 = -5/6, c_2 = 0$, which implies the particular solution $x(t) = -\frac{5}{6}\cos(8t) + \frac{5}{6}\cos(4t)$.

Part (e) The answer is $x(t) = -16\sin(5t) + 20\sin(4t)$ by the method of undetermined coefficients.

Rule I: $x = d_1\cos(4t) + d_2\sin(4t)$. Rule II does not apply due to natural frequency $\sqrt{25} = 5$ not equal to the frequency of the trial solution (no conflict). Substitute the trial solution into $x''(t) + 25x(t) = 180\sin(4t)$ to get $9d_1\cos(4t) + 9d_2\sin(4t) = 180\sin(4t)$. Match coefficients, to arrive at the equations $9d_1 = 0$, $9d_2 = 180$. Then $d_1 = 0$, $d_2 = 20$ and $x_p(t) = 20\sin(4t)$. Lastly, add the homogeneous solution to obtain $x(t) = x_h + x_p = c_1\cos(5t) + c_2\sin(5t) + 20\sin(4t)$. Use the initial condition relations $x(0) = 0, x'(0) = 0$ to obtain the equations $\cos(0)c_1 + \sin(0)c_2 + 20\sin(0) = 0$, $-5\sin(0)c_1 + 5\cos(0)c_2 + 80\cos(0) = 0$. Solve for the coefficients $c_1 = 0$, $c_2 = -16$

Part (f) The answer is $x = \sin t - 2\cos t$ by the method of undetermined coefficients.

Rule I: the trial solution $x(t)$ is a linear combination of the Euler atoms found in $f(x) = 5\sin(t)$. Then $x(t) = d_1\cos(t) + d_2\sin(t)$. Rule II does not apply, because solutions of the homogeneous problem contain negative exponential factors (no conflict). Substitute the trial solution into $x'' + 2x' + 2x = 5\sin(t)$ to get $(-2d_1 + d_2)\sin(t) + (d_1 + 2d_2)\cos(t) = 5\sin(t)$. Match coefficients to find the system of equations $(-2d_1 + d_2) = 5$, $(d_1 + 2d_2) = 0$. Solve for the coefficients $d_1 = -2$, $d_2 = 1$.

Part (g) Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4)$, therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor $5r^2 + 2r + 4$ to obtain roots $(-1 \pm \sqrt{19}i)/5$ and then classify as **under-damped**.

Part (h) The resonant frequency is $\omega = 1/\sqrt{LC} = 1/\sqrt{2/50} = \sqrt{25} = 5$. The solution uses the theory in the textbook, section 3.7, which says that electrical resonance occurs for $\omega = 1/\sqrt{LC}$.

Part (i) The answer is $x(t) = -5\cos(5t) - 4\sin(5t)$ by undetermined coefficients.
Rule I: The trial solution is $x_p(t) = A\cos(5t) + B\sin(5t)$. Rule II: because the homogeneous solution $x_h(t)$ has limit zero at $t = \infty$, then Rule II does not apply (no conflict). Substitute the trial solution into the differential equation. Then $-8A\cos(5t) - 8B\sin(5t) - 10A\sin(5t) + 10B\cos(5t) = 82\sin(5t)$. Matching coefficients of sine and cosine gives the equations $-8A + 10B = 0$, $-10A - 8B = 82$. Solving, $A = -5, B = -4$. Then $x_p(t) = -5\cos(5t) - 4\sin(5t)$ is the unique periodic steady-state solution.

Part (j) The characteristic polynomial is the expansion $(r - 1.2)^4((r + 1)^2 + 1)^3$. Because x^3e^{ax} is an Euler solution atom for the differential equation if and only if $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$ are Euler solution atoms, then the characteristic equation must have roots $1.2, 1.2, 1.2, 1.2$, listing according to multiplicity. Similarly, $x^2e^{-x}\sin(x)$ is an Euler solution atom for the differential equation if and only if $-1 \pm i, -1 \pm i, -1 \pm i$ are roots of the characteristic equation. There is a total of 10 roots with product of the factors $(r - 1)^4((r + 1)^2 + 1)^3$ equal to the 10th degree characteristic polynomial.

Chapter 5 Final Exam Sample Solutions. Presently, there are no solutions available for the two sample problems. If you solve one, then kindly email your solution to post.