Instructions: This in-class exam is 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Laplace Theory: Chapter 7

Laplace Theory Background. Expected without notes or books is the 4-entry Forward Laplace Table and these rules: Linearity, Parts, Shift, $s$-differentiation, Lerch’s Theorem, Final Value Theorem, Existence theory.

1. (Laplace Theory: Differential Equations)
   Solve differential equations (1a), (1b) by Laplace’s method.
   (1a) [40%] Third order $x'''' + x' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$.
   (1b) [60%] Dynamical system $x' = x + y$, $y' = 2x + 6$, $x(0) = 0$, $y(0) = 0$.

2. (Laplace Theory: Backward Table)
   Solve for $f(t)$ in (2a), (2b), (2c), (2d).

   Assumptions. Below, $f(t)$ is of piecewise continuous of exponential order. Expression $u(t)$ denotes the unit step function.

   Credit. Document all steps, e.g., if you cancel $\mathcal{L}$ then cite Lerch’s Theorem. The answer is 25%. The documented steps are 75%. Partial fraction coefficients are expected to be evaluated last.

   (2a) [20%] $\mathcal{L}(f(t)) = \frac{100}{(s^2 + 25)(s^2 + 4)}$
   (2b) [20%] $\mathcal{L}(f(t)) = \frac{s + 1}{s^2 + 4s + 29}$.
   (2c) [30%] $\mathcal{L}(f(t)) = \frac{1}{(s^2 + 2s)(s^2 - 3s)}$
   (2d) [30%] $\frac{d}{ds}\mathcal{L}(f(t))\bigg|_{s \rightarrow (s - 3)} = \frac{d}{ds}\mathcal{L}(u(t - \pi)e^t \cos 25t)$.

3. (Laplace Theory: Forward Table)
   Compute the Laplace transform $\mathcal{L}(f(t))$.
   (3a) [20%] $f(t) = (-t)e^{2t} \sin(3t)$.
   (3b) [30%] $f(t) = e^{-\pi t}g(t)$ and $g(t) = \frac{e^{2t} - e^{-2t}}{t}$.
   (3c) [20%] $\frac{d}{ds}\mathcal{L}(f(t)) = \frac{1}{s^{2019}} + \frac{1}{s^2 + 1}$.
   (3d) [30%] Define $f(t) = e^{-2t}g(t)$ where $g(t) = \frac{e^{2t} - e^{-2t}}{t}$.
Systems of Differential Equations: Chapters 4 and 5

Background.

The **Eigenanalysis Method** for a real $3 \times 3$ matrix $A$ assumes eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. It says that the $3 \times 3$ system $\vec{x}' = A\vec{x}$ has general solution $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + c_3 \vec{v}_3 e^{\lambda_3 t}$.

The **Cayley-Hamilton-Ziebur method** is based upon this result:

Let $A$ be an $n \times n$ real matrix. The components of solution $\vec{u}$ of $\frac{d}{dt} \vec{u}(t) = A\vec{u}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$. Alternatively, $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms: $\vec{u}(t) = \sum_{k=1}^{n} (\text{atom}_k) \vec{d}_k$.

A **Fundamental Matrix** is an $n \times n$ matrix $\Phi(t)$ with columns consisting of independent solutions of $\vec{x}'(t) = A\vec{x}(t)$, where $A$ is an $n \times n$ real matrix. The general solution of $\vec{x}'(t) = A\vec{x}(t)$ is $\vec{x}(t) = \Phi(t)\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, with $|\Phi(0)| \neq 0$ required to establish independence of the columns of $\Phi$.

The **Exponential Matrix**, denoted $e^{At}$, is the unique fundamental matrix $\Psi(t)$ such that $\Psi(0) = I$. Matrix $A$ is an $n \times n$ real matrix. It is known that $e^{At} = \Phi(t)\Phi(0)^{-1}$ for *any* fundamental matrix $\Phi(t)$. Consequently, $\frac{d}{dt} (e^{At}) = Ae^{At}$ and $e^{At}|_{t=0} = I$.

4. **(Systems: Eigenanalysis Method)**

Complete parts ((4a), (4b), (4c)).

(4a) [40%] Let $A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{pmatrix}$. Display the linear algebra details for computing the three eigenpairs of $A$.

(4b) [30%] Matrix $A = \begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix}$ has eigenpairs $2, \begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}$, $3, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$.

(4c) [40%] Let $\vec{u}(t) = \vec{c} e^{rt}$, the vector Euler substitution. Assume real $n \times n$ matrix $A$ has a real eigenpair $(\lambda, \vec{v})$. Prove that the Euler substitution with $r = \lambda$ and $\vec{c} = \vec{v}$ applied to $\frac{d}{dt} \vec{u}(t) = A\vec{u}(t)$ finds a nonzero solution of $\frac{d}{dt} \vec{u}(t) = A\vec{u}(t)$.
5. (Systems: First Order Cayley-Hamilton-Ziebur)

(5a) [30%] The eigenvalues are 2, 6 for the matrix \( A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \).

Display the general solution of \( \frac{d}{dt} \vec{u}(t) = A\vec{u}(t) \) according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5).

(5b) [40%] The 3 × 3 triangular matrix \( A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{pmatrix} \) represents a linear cascade, such as found in brine tank models. Apply the linear integrating factor method and related shortcuts (Ch 1 in the textbook) to find the components \( x_1, x_2, x_3 \) of the vector general solution \( \vec{x}(t) \) of \( \frac{d}{dt} \vec{x}(t) = A\vec{x}(t) \).

(5c) [30%] The Cayley-Hamilton-Ziebur shortcut applies to the system

\[
x' = 3x + 2y, \quad y' = -2x + 3y,
\]

which has complex eigenvalues \( \lambda = 3 \pm 2i \). Find a fundamental matrix \( \Phi(t) \) for this system, documenting all details of the computation.

6. (Systems: Second Order Cayley-Hamilton-Ziebur)

Assume below that real 2 × 2 matrix \( A = \begin{pmatrix} -13 & -6 \\ 6 & -28 \end{pmatrix} \) has eigenpairs \(( -25, \begin{pmatrix} 1 \\ 2 \end{pmatrix} )\), \(( -16, \begin{pmatrix} 2 \\ 1 \end{pmatrix} )\). Textbook theorems applied to \( \frac{d^2}{dt^2} \vec{u}(t) = A\vec{u}(t) \) report general solution

\[
\vec{u}(t) = (c_1 \cos(5t) + c_2 \sin(5t)) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_3 \cos(4t) + c_4 \sin(4t)) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

(6a) [30%] Derive the characteristic equation \( |A - r^2 I| = 0 \) for the second order equation \( \frac{d^2}{dt^2} \vec{u}(t) = A\vec{u}(t) \) from Euler’s vector substitution \( \vec{u}(t) = \vec{c} e^{rt} \). A proof is expected with details.\(^1\)

(6b) [40%] Substitute \( \vec{u}(t) = \vec{d} \cos(5t) \) into \( \frac{d^2}{dt^2} \vec{u}(t) = A\vec{u}(t) \) to determine vector \( \vec{d} \) in terms of eigenpairs of \( A \). Repeat for \( \vec{u}(t) = \vec{d} \sin(5t) \) and report the answer.

(6c) [30%] The Euler solution atoms \( \cos(5t), \sin(5t), \cos(4t), \sin(4t) \) are linearly independent on \(( -\infty, \infty )\). Substitute \( \vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(4t) + \vec{d}_4 \sin(4t) \) into \( \frac{d^2}{dt^2} \vec{u}(t) = A\vec{u}(t) \) and use independence (vector coefficients of atoms match) to determine the vectors \( \vec{d}_1, \vec{d}_2, \vec{d}_3, \vec{d}_4 \) in terms of eigenpairs of \( A \).

\(^1\)Reminder: Linear algebra writes eigenpair equation \( A\vec{x} = \lambda \vec{x} \) equivalently as \( A\vec{x} = \lambda I\vec{x} \) and then converts it to the homogeneous system of linear algebraic equations \( (A - \lambda I)\vec{x} = \vec{0} \). The proof you write should apply without edits to \( n \times n \) real matrices \( A \).