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Differential Equations 2280

Midterm Exam 1

Exam Date: Friday, 16 February 2018 at 12:50pm

Instructions: This in-class exam is designed for 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [40%] Solve $y' = \dfrac{2x^3}{1+x^2}$.

(b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{2t}v(t)) = 2$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 2$.

Solution to Problem 1.

(a) Answer $y = x^2 - \ln\left(x^2+1\right) + c$. Treat the problem as a quadrature problem $y' = F(x)$, then $y = \int F(x)dx$. Integration details:

$$\begin{aligned}
\int F(x)dx &= \int \frac{2x^3}{1+x^2}dx \\
&= \int \frac{x^2}{u}2xdx, \quad u = 1+x^2, du = 2xdx \\
&= \int \frac{u-1}{u}du \\
&= \int (1-1/u)du \\
&= u - \ln|u| + c_1 \\
&= 1 + x^2 - \ln\left|1+x^2\right| + c_1 \\
&= x^2 - \ln\left|1+x^2\right| + c
\end{aligned}$$

(b) Velocity $v(t) = 2t\,e^{-2t} + 5\,e^{-2t}$ by quadrature. Integrate $x'(t) = 2t\,e^{-2t} + 5\,e^{-2t}$ with $x(0) = 2$ to obtain position $x(t) = -(t+3)e^{-2t} + 5$. The integral of te^{-2t} is found using integration by parts. See Exercise 1.2-10 in Edwards-Penney.

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2. (Solve a Separable Equation)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [30%] The equation $y' + x(2y + 3) = ye^{2x} + 3x$ is separable. Provide formulas for the functions F and G.

(b) [70%] Find a non-equilibrium solution in implicit form for the separable equation

$$(5)y' = \left(\frac{2x}{1 + x^2} + \cos(x)\sin(x)\right)(y^2 - 3y + 2)$$

To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 2.

(a) The equation is $y' = ye^{2x} + 3x - x(2y + 3) = (e^{2x} - 2x)y$. Then $F(x) = e^{2x} - 2x$, $G(y) = y$.

(b) The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using definitions

$$F(x) = \frac{2x}{1 + x^2} + \cos(x)\sin(x), \quad G(y) = \frac{y^2 + 3y + 2}{5}.$$

The integral of F is from standard formulas and u-substitution.

$$\begin{aligned}
\textstyle\int F dx &= \textstyle\int \frac{2x}{1 + x^2} + \cos(x)\sin(x)\, dx \\
&= I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
I_1 &= \textstyle\int \frac{2x}{1 + x^2}\, dx \\
&= \textstyle\int \frac{1}{u} 2x dx, \quad u = 1 + x^2, du = 2x dx \\
&= \textstyle\int \frac{1}{u}\, du \\
&= \ln|u| + c_1 \\
&= \ln\left|1 + x^2\right| + c_1 \\
I_2 &= \textstyle\int \cos(x)\sin(x)\, dx \\
&= \textstyle\int u \cos(x) dx, \quad u = \sin(x), du = \cos(x) dx \\
&= \textstyle\int u\, du \\
&= \tfrac{1}{2}u^2 + c_2 \\
&= \tfrac{1}{2}\sin^2(x) + c_2
\end{aligned}$$

Then $\displaystyle\int F(x) dx = \ln\left|1 + x^2\right| + \frac{1}{2}\sin^2(x) + c_3$.

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{aligned}
\int \frac{dx}{G(y(x))} &= \int \frac{5}{u^2+3u+2}\,du, \quad u = y(x), du = y'(x)dx, \\
&= \int \frac{5}{(u+2)(u+1)}\,du \\
&= \int \frac{A}{u+2} + \frac{B}{u+1}\,du, \quad A, B \quad \text{determined later}, \\
&= A\ln|u+2| + B\ln|u+1| + c_4
\end{aligned}$$

The partial fraction problem

$$\frac{5}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

can be solved in a variety of ways, with answer $A = -5$ and $B = 5$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$-5\ln|y+2| + 5\ln|y+1| = \ln\left|1+x^2\right| + \frac{1}{2}\sin^2(x) + c.$$

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3. (Linear Equations)

(a) [60%] Solve the linear model $3x'(t) = -32 + \frac{10}{3t+2}x(t)$, $x(0) = 64$. Show all integrating factor steps.

(b) [20%] Solve $\frac{dy}{dx} + (\tan(x))y = 0$ using the homogeneous linear equation shortcut.

(c) [20%] Solve $7\frac{dy}{dx} = 21y + 19$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

Solution to Problem 3.

(a) The answer is $v(t) = 64 + 96t$. The details:

$$
\begin{aligned}
&v'(t) = -\tfrac{16}{3} + \frac{10/3}{3t+2}v(t),\\
&v'(t) + \frac{-10/3}{3t+2}v(t) = -\frac{32}{3}, \quad \text{standard form } v' + p(t)v = q(t)\\
&p(t) = \frac{-10/3}{3t+2},\\
&W = e^{\int p\,dt}, \quad \text{integrating factor}\\
&W = e^u, \quad u = \int p\,dt = -\tfrac{10/3}{3}\ln|3t+2| = \ln\left(|3t+2|^{-10/9}\right)\\
&W = (3t+2)^{-10/9}, \quad \text{Final choice for } W.
\end{aligned}
$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$
\begin{aligned}
&v'(t) + \frac{-10/3}{3t+2}v(t) = -32/3, \quad \text{standard form } v' + p(t)v = q(t)\\
&\frac{(vW)'}{W} = -\frac{32}{3}, \quad \text{Replace left side by quotient } (vW)'/W\\
&(vW)' = -\tfrac{32}{3}W, \quad \text{cross-multiply}\\
&vW = -\tfrac{32}{3}\int W\,dt, \quad \text{quadrature step.}
\end{aligned}
$$

The evaluation of the integral is from the power rule:

$$\int -\frac{32}{3}W\,dt = -\frac{32}{3}\int (3t+2)^{-10/9}dt = -\frac{32}{3}\frac{(3t+2)^{-1/9}}{(-1/9)(3)} + c.$$

Division by $W = (3t+2)^{-10/9}$ then gives the general solution

$$v(t) = 32(3t+2) + \frac{c}{W}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 64$. Then

$$v(t) = 96t + 64.$$

(b) The answer is $y =$ constant divided by the integrating factor: $y = \frac{c}{W}$. Because $W = e^u$ where $u = \int \sin(x)dx = -\cos x$, then $y = ce^{\cos x}$.

(c) The equilibrium solution (a constant solution) is $y_p = -\frac{7}{21}$. The homogeneous solution is $y_h = ce^{21x/5}$ = constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{1}{3} + c\,e^{21x/5}$.

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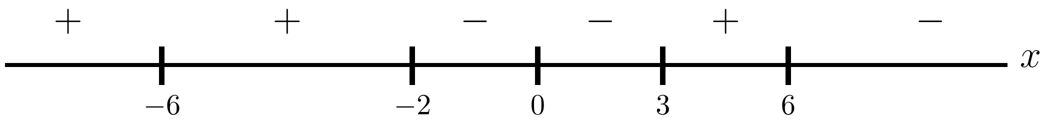
4. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 4.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point. 5.

Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -6$: NODE, UNSTABLE
$x = -2$: FUNNEL, STABLE
$x = 0$: NODE, UNSTABLE
$x = 3$: SPOUT, UNSTABLE
$x = 6$: FUNNEL, STABLE

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5. (ch3)
Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a constant coefficient differential equation $ay''' + by'' + cy' + dy = 0$ which has a particular solution $-15e^{-x} + 3xe^{-x} + 100$.

(b) [30%] Solve the 6th order equation $y^{(6)} + 4y^{(5)} + 4y^{(4)} = 0$.

(c) [30%] Solve the higher order homogeneous linear differential equation whose characteristic equation is $r^2(r+2)(r^3+2r)(r^2+2r+101) = 0$.

Solution to Problem 5.
5(a)
A solution of a constant-coefficient linear homogeneous differential equation is a linear combinations of Euler solution atoms. The given particular solution is a linear combination of Euler atoms e^{-x}, $x\,e^{-x}$, 1. Because $1 = e^{0x}$, then one root of the characteristic equation is $r = 0$. Due to Euler's multiplicity theorem, the Euler atoms e^{-x}, $x\,e^{-x}$ account for a double root $r = -1, -1$. Then the characteristic equation has factors $(r+1)$, $(r+1)$, r [College Algebra Root-Factor Theorem applied]. The characteristic equation is then

$$(r+1)(r+1)r = 0$$

which is $r^3 + 2r^2 + r = 0$. Solving backwards gives the differential equation $y''' + 2y'' + y' = 0$.

The answer is the differential equation $y''' + 2y'' + y' = 0$.

5(b)
The characteristic equation $r^6+4r^5+4r^4 = 0$ factors into $r^4(r^2+4r+4) = 0$ with roots $0, 0, 0, 0, -2, -2$. The solution y is a linear combination of the Euler solution atoms $1, x, x^2, x^3, e^{-2x}, xe^{-2x}$.

5(c)
The characteristic equation factors into $r^3\,(r+2)\left(r^2+2\right)\left(r^2+2\,r+101\right)$ with roots

$$r = 0, 0, 0; -2; i\sqrt{2}; -i\sqrt{2}; -1 \pm 10i.$$

Then y is a linear combination of the Euler solution atoms

$$1, x, x^2, e^{-2x}, \cos(x\sqrt{2}), \sin(x\sqrt{2}), e^{-x}\cos(10x), e^{-x}\sin(10x)$$