Differential Equations 2280

Sample Midterm Exam 1

Exam Date: Friday, 22 February 2019 at 7:30am in LCB 215

Instructions: This in-class exam is designed for 80 minutes. Maximum of seven (7) problems. Expect dramatically fewer parts to each problem. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [40%] Solve $y' = \dfrac{2x^3}{1+x^2}$.

(b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}v(t)) = 2\,e^t$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = 2$.

Solution to Problem 1.

(a) Answer $y = x^2 - \ln\left(x^2+1\right) + c$. The integral of $F(x) = \frac{2x^3}{1+x^2}$ is found by substitution $u = 1+x^2$, resulting in the new integration problem $\int F\,dx = \int \frac{u-1}{u}du = \int (1)du - \int \frac{du}{u}$.

(b) Velocity $v(t) = 2e^{2t} + 3e^t$ by quadrature. Integrate $x'(t) = 2e^{2t} + 3e^t$ with $x(0) = 2$ to obtain position $x(t) = e^{2t} + 3e^t - 2$.

2. (Solve a Separable Equation)

Given $(5y+10)y' = \left(xe^{-x} + \sin(x)\cos(x)\right)\left(y^2+3y-4\right)$.

Find a non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 2.

The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using

$$F(x) = xe^{-x} + \sin(x)\cos(x), \quad G(y) = \frac{y^2+3y-4}{5y+10}.$$

The integral of F is done by parts and also by u-substitution.

$$\begin{aligned}
\textstyle\int F dx &= \textstyle\int xe^{-x}\,dx + \int \sin(x)\cos(x)\,dx \\
&= I_1 + I_2. \\
I_1 &= \textstyle\int xe^{-x}\,dx \\
&= -xe^{-x} - \textstyle\int e^{-x}\,dx, \quad \text{parts} \quad u = x, dv = e^{-x}dx, \\
&= -xe^{-x} - e^{-x} + c_1 \\
I_2 &= \textstyle\int \sin(x)\cos(x)\,dx \\
&= \textstyle\int u\,du, \quad u = \sin(x), du = \cos(x)dx, \\
&= u^2/2 + c_2 \\
&= \tfrac{1}{2}\sin^2(x) + c_2
\end{aligned}$$

Then $\displaystyle\int F dx = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c_3$.

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{aligned}
\int \frac{dx}{G(y(x))} &= \int \frac{5u+10}{u^2+3u-4}\,du, \quad u = y(x), du = y'(x)dx, \\
&= \int \frac{5u+10}{(u+4)(u-1)}\,du \\
&= \int \frac{A}{u+4} + \frac{B}{u-1}\,du, \quad A, B \quad \text{determined later}, \\
&= A\ln|u+4| + B\ln|u-1| + c_4
\end{aligned}$$

The partial fraction problem
$$\frac{5u+10}{(u+4)(u-1)} = \frac{A}{u+4} + \frac{B}{u-1}$$
can be solved in a variety of ways, with answer $A = \frac{-20+10}{-5} = 2$ and $B = \frac{15}{5} = 3$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$2\ln|y+4| + 3\ln|y-1| = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c.$$

3. (Linear Equations)

(a) [60%] Solve the linear model $2x'(t) = -64 + \dfrac{10}{3t+2}x(t)$, $x(0) = 32$. Show all integrating factor steps.
(b) [20%] Solve $\dfrac{dy}{dx} - (\cos(x))y = 0$ using the homogeneous linear equation shortcut.
(c) [20%] Solve $5\dfrac{dy}{dx} - 7y = 10$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

Solution to Problem 3.
(a) The answer is $v(t) = 32 + 48t$. The details:

$$\begin{aligned}
&v'(t) = -32 + \frac{5}{3t+2}v(t),\\
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t)\\
&p(t) = \frac{-5}{3t+2},\\
&W = e^{\int p\,dt}, \quad \text{integrating factor}\\
&W = e^u, \quad u = \textstyle\int p\,dt = -\frac{5}{3}\ln|3t+2| = \ln\left(|3t+2|^{-5/3}\right)\\
&W = (3t+2)^{-5/3}, \quad \text{Final choice for } W.
\end{aligned}$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$\begin{aligned}
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t)\\
&\frac{(vW)'}{W} = -32, \quad \text{Replace left side by quotient } (vW)'/W\\
&(vW)' = -32W, \quad \text{cross-multiply}\\
&vW = -32\textstyle\int W\,dt, \quad \text{quadrature step.}
\end{aligned}$$

The evaluation of the integral is from the power rule:

$$\int -32W\,dt = -32\int(3t+2)^{-5/3}dt = -32\frac{(3t+2)^{-2/3}}{(-2/3)(3)} + c.$$

Division by $W = (3t+2)^{-5/3}$ then gives the general solution

$$v(t) = \frac{c}{W} - \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 32$. Then

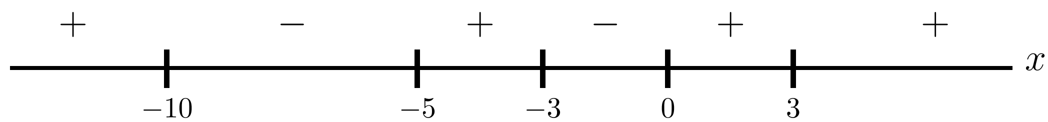
$$v(t) = \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3} = 16(3t+2)^{-\frac{2}{3}+\frac{5}{3}} = 16(3t+2).$$

(b) The answer is $y =$ constant divided by the integrating factor: $y = \dfrac{c}{W}$. Because $W = e^u$ where $u = \int -\cos(x)dx = -\sin x$, then $y = ce^{\sin x}$.

(c) The equilibrium solution (a constant solution) is $y_p = -\frac{10}{7}$. The homogeneous solution is $y_h = ce^{7x/5}$ = constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{10}{7} + ce^{7x/5}$.

4. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 4.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -10$: FUNNEL, STABLE
$x = -5$: SPOUT, UNSTABLE
$x = -3$: FUNNEL, STABLE
$x = 0$: SPOUT, UNSTABLE
$x = 3$: NODE, UNSTABLE

5. (Chapter 3: Linear nth Order DE)
Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a constant coefficient differential equation $ay'' + by' + cy = 0$ which has particular solutions $-5e^{-x} + xe^{-x}$, $10e^{-x} + xe^{-x}$.

(b) [30%] Given characteristic equation $r(r-2)(r^3+4r)(r^2+2r+37) = 0$, solve the differential equation.

(c) [30%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 129$. Classify the equation as over-damped, critically damped or under-damped. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = 1$, $x'(0) = 0$.

Solution to Problem 5.

5(a)

Multiply the first solution by 2 and add it to the second solution. Then Euler atom xe^{-x} is a solution, which implies that $r = -1$ is a double root of the characteristic equation. Then the characteristic equation should be $(r - (-1))(r - (-1)) = 0$, or $r^2 + 2r + 1 = 0$. The differential equation is $y'' + 2y' + y = 0$.

5(b)

The characteristic equation factors into $r^2(r - 2)(r^2 + 4)((r + 1)^2 + 36) = 0$ with roots $r = 0, 0; 2; \pm 2i; -1 \pm 6i$. Then y is a linear combination of the Euler solution atoms:

$1, x, e^{2x}, \cos(2x), \sin(2x); e^{-x}\cos(6x), e^{-x}\sin(6x)$.

5(c)

Use $4r^2 + 4r + 129 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4\sqrt{2}i, -1/2 - 4\sqrt{2}i$ and Euler solution atoms $e^{-x/2}\cos 4\sqrt{2}t, e^{-x/2}\sin 4\sqrt{2}t$. Then y is a linear combination of these two solution atoms, and it oscillates, therefore the classification is **under-damped**. The illustration shows a spring, a dashpot and a mass with labels k, c, m. Initial conditions mean mass elongation $x = 1$, at rest.

A **dashpot** is represented as a cylinder and piston with rod, the rod attached to the mass. Variable x is positive in the down direction and negative in the up direction. The equilibrium position is $x = 0$.

6. (Chapter 3: Linear nth Order DE)

(a) [25%] The trial solution y with fewest Euler solution atoms, according to the method of undetermined coefficients, contains no solution of the homogeneous equation. Explain why, using the example $y'' = 1 + x$.

(b) [75%] Determine for $y^{(3)} + y^{(2)} = x + 2e^{-x} + \sin x$ the corrected trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate the undetermined coefficients**! The trial solution should be the one with fewest Euler solution atoms.

Solution to Problem 6.

(a). Rule I says the trial solution is $y = d_1 + d_2x$. Rule II says to multiply by x until no atom is a solution of $y'' = 0$. Then $y = d_1x^2 + d_2x^3$ contains no terms of the homogeneous solution $y_h = c_1 + c_2x$.

(b). The homogeneous equation $y^{(3)} + y^{(2)} = 0$ has solution $y_h = c_1 + c_2x + c_3e^{-x}$, because the characteristic polynomial has roots 0, 0, -1.

$\boxed{\mathbf{1}}$ Rule I constructs an initial trial solution y from the list of independent Euler solution atoms

$$e^{-x}, \quad 1, \quad x, \quad \cos x, \quad \sin x.$$

Linear combinations of these atoms are supposed to reproduce, by assignment of constants, all derivatives of $f(x) = x + 2e^{-x} + \sin x$, which is the right side of the differential equation. Each of y_1 to y_4 in the display below is constructed to have the same **base atom**, which is the Euler atom obtained by stripping the power of x. For example, Euler solution atom xe^{0x} (or x, because $e^{0x} = 1$) strips to base atom e^{0x} or 1.

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1e^{-x}, \\ y_2 &= d_2 + d_3x, \\ y_3 &= d_4\cos x, \\ y_4 &= d_5\sin x. \end{aligned}$$

$\boxed{\mathbf{2}}$ Rule II is applied individually to each of y_1, y_2, y_3, y_4 to give the **corrected trial solution**

$$\begin{aligned} y &= y_1 + y_2 + y_3 + y_4, \\ y_1 &= d_1xe^{-x}, \\ y_2 &= x^2(d_2 + d_3x), \\ y_3 &= d_4\cos x, \\ y_4 &= d_5\sin x. \end{aligned}$$

The powers of x multiplied in each case are selected to eliminate terms in the initial trial solution which duplicate homogeneous equation Euler solution atoms. For instance, $y_1 = d_1 e^{-x}$ is **in conflict** with the homogeneous equation, because e^{-x} is a common Euler atom of both y_1 and the homogeneous solution ($y_h = c_1 + c_2 x + c_3 e^{-x}$). Then Rule II multiplies y_1 by x to obtain the replacement $y_1 = d_1 x e^{-x}$. This new term is again subjected to the Rule II test: Does y_1 contain an Euler atom of the homogeneous equation? The answer is NO, so the x-multiplication stops and the term y_1 is finished. We go on to the remaining terms, in the same way. Term y_2 needs two x-multiplications. The factor used after so many x-multiplications is exactly x^s of the Edwards-Penney table, where s is the multiplicity of the characteristic equation root r that produced the conflicting atom in the homogeneous solution y_h. The atoms in terms y_3, y_4 are not solutions of the homogeneous equation, therefore y_3, y_4 are unaltered by Rule II.

7. (Chapters 1 and 3)

(a) [20%] Solve $2v'(t) = -8 + \dfrac{2}{2t+1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

(b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

(c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2 + r)^2(r^2 + 9)^2 = 0$.

(d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$.

(e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0\cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

(f) [25%] Determine for $y''' + y'' = 100x^2 + \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

(g) [50%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = -10\sin(t)$.

(h) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.

Solution to Problem 7.

(a) $v(t) = -4 - 8t$

(b) $r^2 + 4r + 6 = 0$, $y = c_1 y_1 + c_2 y_2$, $y_1 = e^{-2x}\cos\sqrt{2}x$, $y_2 = e^{-2x}\sin\sqrt{2}x$.

(c) Write as $r^3(r+1)^2(r^2+9)^2 = 0$. Then y is a linear combination of the atoms 1, x, x^2, e^{-x}, xe^{-x}, $\cos 3x$, $x\cos 3x$, $\sin 3x$, $x\sin 3x$.

(d) The atoms that appear in $y(x)$ are $x, \sin\sqrt{2}x, e^{-x}\cos 3x$. Derivatives of these atoms create a longer list: $1, x, \cos\sqrt{2}x, \sin\sqrt{2}x, e^{-x}\cos 3x, e^{-x}\sin 3x$. These atoms correspond to characteristic equation roots $0, 0$; $\sqrt{2}i$, $-\sqrt{2}i$, $-1+3i$, $-1-3i$. Then the characteristic equation has factors r, r; $x^2 + 2$; $((r+1)^2 + 9)$. The product of these factors is the correct characteristic equation, which corresponds to the differential equation of least order such that $y(x)$ is a solution. Report $r^6 + 2r^5 + 12r^4 + 4r^3 + 20r^2 = 0$ as the characteristic equation or $y^{(6)} + 2y^{(5)} + 12y^{(4)} + 4y''' + 20y'' = 0$ as the differential equation.

(e) It has to equal the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then $x_{SS}(t) = 11\cos 2t - \sqrt{11}\sin 2t$.

(f) The homogeneous solution is a linear combination of the atoms 1, x, e^{-x} because the characteristic polynomial has roots 0, 0, -1.

Rule $\boxed{1}$ An initial trial solution y is constructed for atoms 1, x, x^2, $\cos x$, $\sin x$ giving 3 groups, each group having the same base atom:

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= d_1 + d_2 x + d_3 x^2, \\ y_2 &= d_4\cos x, \\ y_3 &= d_5\sin x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.
Rule $\boxed{2}$ The correction rule is applied individually to each of y_1, y_2, y_3.
Multiplication by x is done repeatedly, until the replacement atoms do not appear in atom list for the homogeneous differential equation. The result is the **shortest trial solution**

$$y = y_1 + y_2 + y_3 = (d_1x^2 + d_2x^3 + d_3x^4) + (d_4 \cos x) + (d_5 \sin x).$$

Some facts: (1) If an Euler solution atom of the homogeneous equation appears in a group, then it is removed because of x-multiplication, but replaced by a new atom having the same base atom. (2) The number of terms in each of y_1 to y_3 is unchanged from Rule I to Rule II.

(g) Answer: $I_{\rm SS}(t) = \cos t - 2\sin t$.

Variation of Parameters.
This is the same problem as $x'' + 2x' + 5x = -10\sin(t)$.
Solve $x'' + 2x' + 5x = 0$ to get $x_h = c_1x_1 + c_2x_2$, $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = 4e^{-2t}$. Then for $f(t) = -10\sin(t)$,

$$x_p = x_1 \int x_2 \frac{-f}{W} dt + x_2 \int x_1 \frac{f}{W} dt.$$

The integrations are too difficult, so the method won't be pursued.
Undetermined Coefficients.
The trial solution by Rule I is $I = d_1 \cos t + d_2 \sin t$. The homogeneous solutions have exponential factors, therefore the Euler solution atoms in the trial solution cannot be solutions of the homogeneous problem, hence Rule II does not apply.
Substitute the trial solution into the non-homogeneous equation to obtain the answers $d_1 = 1$, $d_2 = -2$. The unique periodic solution $I_{\rm SS}$ is extracted from the general solution $I = I_h + I_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $I_p = d_1 \cos t + d_2 \sin t = \cos t - 2\sin t$ and the homogeneous solution x_h has negative exponential terms, then

$$I_{\rm SS} = \cos t - 2\sin t.$$

Laplace Theory.
Plan: Find the general solution, then extract the steady-state solution by dropping negative exponential terms. The computation can use initial data $I(0) = I'(0) = 0$, because every particular solution contains the terms of the steady-state solution. Some details:

$$(s^2 + 2s + 5)\mathcal{L}(I) = \frac{-10}{s^2+1}$$
$$\mathcal{L}(I) = \frac{-10}{(s^2+1)(s^2+2s+5)}$$
$$\mathcal{L}(I) = \frac{-10}{(s^2+1)((s+1)^2+4)}$$
$$\mathcal{L}(I) = \frac{s-2}{s^2+1} - \frac{s}{(s+1)^2+4}$$
$$\mathcal{L}(I) = \frac{s}{s^2+1} - 2\frac{1}{s^2+1} - \frac{s+1}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}$$
$$\mathcal{L}(I) = \mathcal{L}(\cos t) - 2\mathcal{L}(\sin t) - \mathcal{L}(e^{-t}\cos 2t) + \frac{1}{2}\mathcal{L}(e^{-t}\sin 2t)$$

$I(t) = \cos t - 2\sin t - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$, by Lerch's Theorem.
Dropping the negative exponential terms gives the steady-state solution $I_{\rm SS}(t) = \cos t - 2\sin t$.

(h) Answer: $y_p = \dfrac{x^2}{2} - \dfrac{x^3}{6}$.

Variation of Parameters.
Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W =$

$y_1 y_2' - y_1' y_2 = 1$. Then for $f(t) = 1 - x$,
$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx,$
$y_p = 1 \int -x(1-x) dx + x \int 1(1-x) dx,$
$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2),$
$y_p = x^2/2 - x^3/6.$
This answer is checked by quadrature, applied twice to $y'' = 1 - x$ with initial conditions zero.