11.8 Second-order Systems

A model problem for second order systems is the system of three masses coupled by springs studied in section 11.1, equation (6):

$$\text{(1)}\qquad \begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\ m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\ m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t). \end{aligned}$$

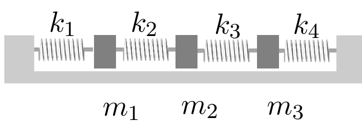

Figure 22. Three masses connected by springs. The masses slide on a frictionless surface.

In vector-matrix form, this system is a **second order system**

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t)$$

where the **displacement** $\vec{\mathbf{x}}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \; M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \; K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Because M is invertible, the system can always be written as

$$\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}, \quad A = M^{-1}K.$$

Converting $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ to $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$

Given a second order $n \times n$ system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, define the variable $\vec{\mathbf{u}}$ and the $2n \times 2n$ block matrix C as follows.

$$\text{(2)}\qquad \vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{pmatrix}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right).$$

Then each solution $\vec{\mathbf{x}}$ of the second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ produces a corresponding solution $\vec{\mathbf{u}}$ of the first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. Similarly, each solution $\vec{\mathbf{u}}$ of $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ gives a solution $\vec{\mathbf{x}}$ of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by the formula $\vec{\mathbf{x}} = \textbf{diag}(I, 0)\vec{\mathbf{u}}$.

Euler's Substitution $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$

The fundamental substitution of L. Euler applies to vector-matrix differential systems. In particular, for $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, the equation $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ produces the **characteristic equation**

$$\det(A - \lambda^2 I) = 0,$$

and the **eigenpair equation**

$$A\vec{\mathbf{v}} = \lambda^2\vec{\mathbf{v}}, \quad \vec{\mathbf{v}} \neq \vec{\mathbf{0}},$$

which means that $(\lambda^2, \vec{\mathbf{v}})$ is an eigenpair of the matrix A.

Negative eigenvalues of A produce complex conjugate values for λ. For instance, $\lambda^2 = -4$ implies $\lambda = \pm 2i$, and then, even though vector $\vec{\mathbf{v}}$ has real components, the solution $\vec{\mathbf{x}}(t) = e^{\lambda t}\vec{\mathbf{v}}$ is a vector with complex entries: $\vec{\mathbf{x}}(t) = e^{2it}\vec{\mathbf{v}} = \cos(2t)\vec{\mathbf{v}} + i\sin(2t)\vec{\mathbf{v}}$.

Details. Compute $\vec{\mathbf{x}}' = \frac{d}{dt}e^{\lambda t}\vec{\mathbf{v}} = \lambda e^{\lambda t}\vec{\mathbf{v}} = \lambda\vec{\mathbf{x}}$. Then $\vec{\mathbf{x}}'' = \lambda^2\vec{\mathbf{x}}$. If $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, then $\lambda^2\vec{\mathbf{x}} = A\vec{\mathbf{x}}$ holds, which is equivalent to $\lambda^2\vec{\mathbf{v}} = A\vec{\mathbf{v}}$. Then $(\lambda^2, \vec{\mathbf{v}})$ is an eigenpair of A. Conversely, if $(\lambda^2, \vec{\mathbf{v}})$ is an eigenpair of A, then the steps reverse to obtain $\lambda^2\vec{\mathbf{x}} = A\vec{\mathbf{x}}$, which means that $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

By linear algebra, the equation $A\vec{\mathbf{v}} = \lambda^2\vec{\mathbf{v}}$ has a solution $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ if and only if the homogeneous problem $(A - \lambda^2 I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has infinitely many solutions. Cramer's Rule implies this event happens exactly when $\det(A - \lambda^2 I) = 0$.

Characteristic Equation for $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

The characteristic equation for the $n \times n$ second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ will be derived anew from the corresponding $2n \times 2n$ first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. We will prove the following identity.

Theorem 31 (Characteristic Equation)
Let $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ be given with $n \times n$ constant matrix A. Let $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ be its corresponding first order system, where

$$\vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{pmatrix}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right).$$

Then

$$\det(C - \lambda I) = (-1)^n \det(A - \lambda^2 I). \tag{3}$$

Proof: The method of proof is to verify the product formula

$$\left(\begin{array}{r|r} -\lambda I & I \\ \hline A & -\lambda I \end{array}\right)\left(\begin{array}{r|r} I & 0 \\ \hline \lambda I & I \end{array}\right) = \left(\begin{array}{r|r} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right).$$

Then the determinant product formula applies to give

$$\det(C - \lambda I)\det\left(\begin{array}{r|r} I & 0 \\ \hline \lambda I & I \end{array}\right) = \det\left(\begin{array}{r|r} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right). \tag{4}$$

Cofactor expansion is applied to give the two identities

$$\det\left(\begin{array}{r|r} I & 0 \\ \hline \lambda I & I \end{array}\right) = 1, \quad \det\left(\begin{array}{r|r} 0 & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right) = (-1)^n \det(A - \lambda^2 I).$$

Then (4) implies (3). The proof is complete.

Solving $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ and $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

Consider the $n \times n$ second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ and its corresponding $2n \times 2n$ first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$, where

$$(5) \qquad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right), \quad \vec{\mathbf{u}} = \left(\begin{array}{c} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{array}\right).$$

Theorem 32 (Eigenanalysis of A and C)
Let A be a given $n\times n$ constant matrix and define the corresponding $2n\times 2n$ system by

$$\vec{\mathbf{u}}' = C\vec{\mathbf{u}}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right), \quad \vec{\mathbf{u}} = \left(\begin{array}{c} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{array}\right).$$

Then

$$(6) \qquad (C - \lambda I)\left(\begin{array}{c} \vec{\mathbf{w}} \\ \vec{\mathbf{z}} \end{array}\right) = \vec{\mathbf{0}} \quad \text{if and only if} \quad \left\{\begin{array}{rcl} A\vec{\mathbf{w}} & = & \lambda^2\vec{\mathbf{w}}, \\ \vec{\mathbf{z}} & = & \lambda\vec{\mathbf{w}}. \end{array}\right.$$

Proof: The result is obtained by block multiplication, because

$$C - \lambda I = \left(\begin{array}{c|c} -\lambda I & I \\ \hline A & -\lambda I \end{array}\right).$$

Theorem 33 (General Solutions of $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ and $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$)
Let A be a given $n\times n$ constant matrix and define the corresponding $2n\times 2n$ system by

$$\vec{\mathbf{u}}' = C\vec{\mathbf{u}}, \quad C = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right), \quad \vec{\mathbf{u}} = \left(\begin{array}{c} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{array}\right).$$

Assume C has eigenpairs $\{(\lambda_j, \vec{\mathbf{y}}_j)\}_{j=1}^{2n}$ and $\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_{2n}$ are independent. Let I denote the $n\times n$ identity and define $\vec{\mathbf{w}}_j = \mathbf{diag}(I, 0)\vec{\mathbf{y}}_j$, $j = 1, \ldots, 2n$. Then $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ and $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ have general solutions

$$\begin{array}{rcll} \vec{\mathbf{u}}(t) & = & c_1 e^{\lambda_1 t}\vec{\mathbf{y}}_1 + \cdots + c_{2n}e^{\lambda_{2n}t}\vec{\mathbf{y}}_{2n} & \quad (2n \times 1), \\ \vec{\mathbf{x}}(t) & = & c_1 e^{\lambda_1 t}\vec{\mathbf{w}}_1 + \cdots + c_{2n}e^{\lambda_{2n}t}\vec{\mathbf{w}}_{2n} & \quad (n \times 1). \end{array}$$

Proof: Let $\vec{\mathbf{x}}_j(t) = e^{\lambda_j t}\vec{\mathbf{w}}_j$, $j = 1, \ldots, 2n$. Then $\vec{\mathbf{x}}_j$ is a solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, because $\vec{\mathbf{x}}_j''(t) = e^{\lambda_j t}(\lambda_j)^2\vec{\mathbf{w}}_j = A\vec{\mathbf{x}}_j(t)$, by Theorem 32. To be verified is the independence of the solutions $\{\vec{\mathbf{x}}_j\}_{j=1}^{2n}$. Let $\vec{\mathbf{z}}_j = \lambda_j\vec{\mathbf{w}}_j$ and apply Theorem 32 to write $\vec{\mathbf{y}}_j = \left(\begin{array}{c} \vec{\mathbf{w}}_j \\ \vec{\mathbf{z}}_j \end{array}\right)$, $A\vec{\mathbf{w}}_j = \lambda_j^2\vec{\mathbf{w}}_j$. Suppose constants $a_1, \ldots, a_{2n}$ are given such that $\sum_{j=1}^{2n} a_k\vec{\mathbf{x}}_j = 0$. Differentiate this relation to give $\sum_{j=1}^{2n} a_k e^{\lambda_j t}\vec{\mathbf{z}}_j = 0$ for all t. Set $t = 0$ in the last summation and combine to obtain $\sum_{j=1}^{2n} a_k\vec{\mathbf{y}}_j = 0$. Independence of $\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_{2n}$ implies that $a_1 = \cdots = a_{2n} = 0$. The proof is complete.

Eigenanalysis when A has Negative Eigenvalues. If all eigenvalues μ of A are negative or zero, then, for some $\omega \geq 0$, eigenvalue μ is related to an eigenvalue λ of C by the relation $\mu = -\omega^2 = \lambda^2$. Then $\lambda = \pm\omega i$ and $\omega = \sqrt{|\mu|}$. Consider an eigenpair $(-\omega^2, \vec{\mathbf{v}})$ of the real $n \times n$ matrix A with $\omega \geq 0$ and let

$$u(t) = \begin{cases} c_1 \cos \omega t + c_2 \sin \omega t & \omega > 0, \\ c_1 + c_2 t & \omega = 0. \end{cases}$$

Then $u''(t) = -\omega^2 u(t)$ (both sides are zero for $\omega = 0$). It follows that $\vec{\mathbf{x}}(t) = u(t)\vec{\mathbf{v}}$ satisfies $\vec{\mathbf{x}}''(t) = -\omega^2 \vec{\mathbf{x}}(t)$ and $A\vec{\mathbf{x}}(t) = u(t)A\vec{\mathbf{v}} = -\omega^2 \vec{\mathbf{x}}(t)$. Therefore, $\vec{\mathbf{x}}(t) = u(t)\vec{\mathbf{v}}$ satisfies $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$.

Theorem 34 (Eigenanalysis Solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$)
Let the $n \times n$ real matrix A have eigenpairs $\{(\mu_j, \vec{\mathbf{v}}_j)\}_{j=1}^n$. Assume $\mu_j = -\omega_j^2$ with $\omega_j \geq 0$, $j = 1, \ldots, n$. Assume that $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ are linearly independent. Then the general solution of $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ is given in terms of $2n$ arbitrary constants $a_1, \ldots, a_n, b_1, \ldots, b_n$ by the formula

$$\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} \left(a_j \cos \omega_j t + b_j \frac{\sin \omega_j t}{\omega_j} \right) \vec{\mathbf{v}}_j \tag{7}$$

This expression uses the limit convention $\left.\dfrac{\sin \omega t}{\omega}\right|_{\omega=0} = t$.

Proof: The text preceding the theorem and superposition establish that $\vec{\mathbf{x}}(t)$ is a solution. It only remains to prove that it is the general solution, meaning that the arbitrary constants can be assigned to allow any possible initial condition $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$, $\vec{\mathbf{x}}'(0) = \vec{\mathbf{y}}_0$. Define the constants uniquely by the relations

$$\begin{aligned} \vec{\mathbf{x}}_0 &= \textstyle\sum_{j=1}^n a_j \vec{\mathbf{v}}_j, \\ \vec{\mathbf{y}}_0 &= \textstyle\sum_{j=1}^n b_j \vec{\mathbf{v}}_j, \end{aligned}$$

which is possible by the assumed independence of the vectors $\{\vec{\mathbf{v}}_j\}_{j=1}^n$. Then equation (7) implies $\vec{\mathbf{x}}(0) = \sum_{j=1}^n a_j \vec{\mathbf{v}}_j = \vec{\mathbf{x}}_0$ and $\vec{\mathbf{x}}'(0) = \sum_{j=1}^n b_j \vec{\mathbf{v}}_j = \vec{\mathbf{y}}_0$. The proof is complete.