11.5 The Eigenanalysis Method

The general solution $\vec{\mathbf{x}}(t) = e^{At}\,\vec{\mathbf{x}}(0)$ of the linear system

$$\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$$

can be obtained entirely by eigenanalysis of the matrix A, which involves finding all eigenpairs. The expected case is when the $n \times n$ matrix A has n independent eigenvectors in its list of eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2), \quad \ldots, \quad (\lambda_n, \vec{\mathbf{v}}_n).$$

It is not required that the eigenvalues $\lambda_1, \ldots, \lambda_n$ be distinct. The eigenvalues can be real or complex.

The Eigenanalysis Method for a 2×2 Matrix

Suppose that A is 2×2 real and has eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2),$$

with $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ independent. The eigenvalues λ_1, λ_2 can be both real. Also, they can be a complex conjugate pair $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$.

It will be shown that the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ can be written as

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2.$$

The details:

$$\begin{aligned}
\vec{\mathbf{x}}' &= c_1(e^{\lambda_1 t})'\vec{\mathbf{v}}_1 + c_2(e^{\lambda_2 t})'\vec{\mathbf{v}}_2 && \text{Differentiate the formula for } \vec{\mathbf{x}}. \\
&= c_1 e^{\lambda_1 t}\lambda_1\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\lambda_2\vec{\mathbf{v}}_2 && \\
&= c_1 e^{\lambda_1 t}A\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}A\vec{\mathbf{v}}_2 && \text{Use } \lambda_1\vec{\mathbf{v}}_1 = A\vec{\mathbf{v}}_1,\ \lambda_2\vec{\mathbf{v}}_2 = A\vec{\mathbf{v}}_2. \\
&= A\left(c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2\right) && \text{Factor } A \text{ left.} \\
&= A\vec{\mathbf{x}} && \text{Definition of } \vec{\mathbf{x}}.
\end{aligned}$$

Let's rewrite the solution $\vec{\mathbf{x}}$ in the vector-matrix form

$$\vec{\mathbf{x}}(t) = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Because eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are assumed independent, then $\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle$ is invertible and setting $t = 0$ in the previous display gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle^{-1}\vec{\mathbf{x}}(0).$$

Because c_1, c_2 can be chosen to produce any initial condition $\vec{\mathbf{x}}(0)$, then $\vec{\mathbf{x}}(t)$ is the *general solution* of the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

The general solution expressed as $\vec{\mathbf{x}}(t) = e^{At}\,\vec{\mathbf{x}}(0)$ leads to the exponential matrix relation

$$e^{At} = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle^{-1}.$$

The formula is immediately useful when the eigenpairs are real.

Complex conjugate eigenvalues. Assume $\lambda_2 = \overline{\lambda}_1$ and λ_1 not real. Eigenpair $(\lambda_2, \vec{\mathbf{v}}_2)$ is never computed or used, because $A\vec{\mathbf{v}}_1 = \lambda_1 \vec{\mathbf{v}}_1$ implies $A\overline{\vec{\mathbf{v}}}_1 = \overline{\lambda}_1 \overline{\vec{\mathbf{v}}}_1$, which implies λ_2 $(= \overline{\lambda}_1)$ has eigenvector $\vec{\mathbf{v}}_2 = \overline{\vec{\mathbf{v}}}_1$.

If A is real, then e^{At} is real, and taking real parts across the formula for e^{At} will give a real formula. Due to the unpleasantness of the complex algebra, we will report the answer found, which is *real*, and then justify it with minimal use of complex numbers.

Define for eigenpair $(\lambda_1, \vec{\mathbf{v}}_1)$ symbols a, b, P as follows:

$$\lambda_1 = a + ib, \quad b > 0, \quad P = \langle\, \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) \rangle.$$

Then

(1) $$e^{At} = e^{at} P \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} P^{-1}.$$

Justification of (1). The formula is established by showing that the matrix $\Phi(t)$ on the right satisfies $\Phi(0) = I$ and $\Phi' = A\Phi$. Then by definition, $e^{At} = \Phi(t)$. For exposition, let

$$R(t) = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \Phi(t) = PR(t)P^{-1}.$$

The identity $\Phi(0) = I$ verified as follows.

$$\begin{aligned} \Phi(0) &= PR(0)P^{-1} \\ &= Pe^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} \\ &= I \end{aligned}$$

Write $\lambda_1 = a + ib$ and $\vec{\mathbf{v}}_1 = \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)$. The expansion of eigenpair relation $A\vec{\mathbf{v}}_1 = \lambda_1 \vec{\mathbf{v}}_1$ into real and imaginary parts gives the relation

$$A\left(\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\right) = (a + ib)\left(\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\right),$$

which shows that

$$AP = P \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Then

$$\begin{aligned}\Phi'(t)\Phi^{-1}(t) &= PR'(t)P^{-1}PR^{-1}(t)P^{-1}\\ &= PR'(t)R^{-1}(t)P^{-1}\\ &= P\left(aI+\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\right)P^{-1}\\ &= P\begin{pmatrix}a & b\\ -b & a\end{pmatrix}P^{-1}\\ &= A\end{aligned}$$

The proof of $\Phi'(t) = A\Phi(t)$ is complete.

The formula for e^{At} implies that the general solution in this special case is

$$\vec{\mathbf{x}}(t) = e^{at}\langle\,\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1)|\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\rangle\begin{pmatrix}\cos bt & \sin bt\\ -\sin bt & \cos bt\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}.$$

The values c_1, c_2 are related to the initial condition $\vec{\mathbf{x}}(0)$ by the matrix identity

$$\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \langle\,\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1)|\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1))^{-1}\vec{\mathbf{x}}(0\rangle.$$

The Eigenanalysis Method for a 3×3 Matrix

Suppose that A is 3×3 real and has eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2), \quad (\lambda_3, \vec{\mathbf{v}}_3),$$

with $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ independent. The eigenvalues λ_1, λ_2, λ_3 can be all real. Also, there can be one real eigenvalue λ_3 and a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$.

The general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ can be written as

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3.$$

The details, which parallel the 2×2 details, are left as an exercise for the reader.

The solution $\vec{\mathbf{x}}$ is written in vector-matrix form

$$\vec{\mathbf{x}}(t) = \langle\vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\rangle\begin{pmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & e^{\lambda_2 t} & 0\\ 0 & 0 & e^{\lambda_3 t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}.$$

Because the three eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are assumed independent, then $\langle\vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2|\vec{\mathbf{v}}_3\rangle$ is invertible. Setting $t = 0$ in the previous display gives

$$\begin{pmatrix}c_1\\ c_2\\ c_2\end{pmatrix} = \langle\vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2|\vec{\mathbf{v}}_3\rangle^{-1}\vec{\mathbf{x}}(0).$$

Constants c_1, c_2, c_3 can be chosen to produce any initial condition $\vec{\mathbf{x}}(0)$, therefore $\vec{\mathbf{x}}(t)$ is the *general solution* of the 3×3 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. There is a corresponding exponential matrix relation

$$e^{At} = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle^{-1}.$$

This formula is normally used when the eigenpairs are real. When there is a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2 = a + ib$, $b > 0$, then as was shown in the 2×2 case it is possible to extract a real solution $\vec{\mathbf{x}}$ from the complex formula and report a real form for the exponential matrix:

$$\begin{aligned} e^{At} &= P \begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt & 0 \\ -e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1}, \\ P &= \langle\, \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) | \vec{\mathbf{v}}_3 \rangle. \end{aligned}$$

The Eigenanalysis Method for an $n\times n$ Matrix

The general solution formula and the formula for e^{At} generalize easily from the 2×2 and 3×3 cases to the general case of an $n\times n$ matrix.

Theorem 17 (The Eigenanalysis Method)
Let the $n\times n$ real matrix A have eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2), \quad \ldots, \quad (\lambda_n, \vec{\mathbf{v}}_n),$$

with n independent eigenvectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$. Then the general solution of the linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is given by

$$\vec{\mathbf{x}}(t) = c_1 \vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t} + \cdots + c_n \vec{\mathbf{v}}_n e^{\lambda_n t}.$$

The vector-matrix form of the general solution is

$$\vec{\mathbf{x}}(t) = \langle \vec{\mathbf{v}}_1 | \cdots | \vec{\mathbf{v}}_n \rangle\, \mathbf{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

This form is real provided all eigenvalues are real. A real form can be made from a complex form by following the example of a 3×3 matrix A. The plan is to list all complex eigenvalues first, in pairs, λ_1, $\overline{\lambda}_1$, ..., λ_p, $\overline{\lambda}_p$. Then the real eigenvalues $r_1, \ldots, r_q$ are listed, $2p + q = n$. Define

$$P = \langle\, \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) | \ldots | \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_{2p-1}) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_{2p-1}) | \vec{\mathbf{v}}_{2p+1} | \cdots | \vec{\mathbf{v}}_n \rangle,$$

$$R_\lambda(t) = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \text{where} \quad \lambda + a + ib, \quad b > 0.$$

Then the real vector-matrix form of the general solution is

$$\vec{\mathbf{x}}(t) = P\,\mathbf{diag}(R_{\lambda_1}(t), \ldots, R_{\lambda_p}(t), e^{r_1 t}, \ldots, e^{r_q t})\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

and

$$e^{At} = P\,\mathbf{diag}(R_{\lambda_1}(t), \ldots, R_{\lambda_p}(t), e^{r_1 t}, \ldots, e^{r_q t})P^{-1}.$$

Remark on Euler Atoms. If the characteristic equation is $(\lambda-1)^3 = 0$ and there are three independent eigenvectors, then the general solution $\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3$ contains no terms with te^t nor t^2e^t. Our intuition from $(\lambda-1)^3 = 0$ is that solution components should be linear combinations of e^t, te^t, t^2e^t. How is that possible? The answer is contained in the linear combination $2e^t + 0te^t + 0t^2e^t$: it is indeed a linear combination of the Euler atoms.

Spectral Theory Methods

The simplicity of Putzer's spectral method for computing e^{At} is appreciated, but we also recognize that the literature has an algorithm to compute e^{At}, devoid of differential equations, which is of fundamental importance in linear algebra. The parallel algorithm computes e^{At} directly from the eigenvalues λ_j of A and certain products of the nilpotent matrices $A - \lambda_j I$. Called **spectral formulas**, they can be implemented in a numerical laboratory or computer algebra system, in order to efficiently compute e^{At}, even in the case of multiple eigenvalues.

Theorem 18 (Computing e^{At} for Simple Eigenvalues)
Let the $n \times n$ matrix A have n simple eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly complex) and define constant matrices $\vec{\mathbf{Q}}_1, \ldots, \vec{\mathbf{Q}}_n$ by the formulas

$$\vec{\mathbf{Q}}_j = \Pi_{i\neq j}\frac{A - \lambda_i I}{\lambda_j - \lambda_i}, \quad j = 1, \ldots, n.$$

Then

$$e^{At} = e^{\lambda_1 t}\vec{\mathbf{Q}}_1 + \cdots + e^{\lambda_n t}\vec{\mathbf{Q}}_n.$$

Theorem 19 (Computing e^{At} for Multiple Eigenvalues)
Let the $n \times n$ matrix A have k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of algebraic multiplicities $m_1, \ldots, m_k$. Let $p(\lambda) = \det(A - \lambda I)$ and define polynomials $a_1(\lambda), \ldots, a_k(\lambda)$ by the partial fraction identity

$$\frac{1}{p(\lambda)} = \frac{a_1(\lambda)}{(\lambda-\lambda_1)^{m_1}} + \cdots + \frac{a_k(\lambda)}{(\lambda-\lambda_k)^{m_k}}.$$

Define constant matrices $\vec{\mathbf{Q}}_1, \ldots, \vec{\mathbf{Q}}_k$ by the formulas

$$\vec{\mathbf{Q}}_j = a_j(A)\Pi_{i\neq j}(A-\lambda_i I)^{m_i}, \quad j = 1,\ldots,k.$$

Then

$$e^{At} = \sum_{i=1}^{k} e^{\lambda_i t}\, \vec{\mathbf{Q}}_i \sum_{j=0}^{m_i-1} (A-\lambda_i I)^j \,\frac{t^j}{j!}. \tag{2}$$

Proof: Let $\vec{\mathbf{N}}_i = \vec{\mathbf{Q}}_i(A-\lambda_i I)$, $1 \le i \le k$. We first prove

Lemma 1 (Properties)

1. $\vec{\mathbf{Q}}_1 + \cdots + \vec{\mathbf{Q}}_k = I$,
2. $\vec{\mathbf{Q}}_i\vec{\mathbf{Q}}_i = \vec{\mathbf{Q}}_i$,
3. $\vec{\mathbf{Q}}_i\vec{\mathbf{Q}}_j = \vec{\mathbf{0}}$ for $i \neq j$,
4. $\vec{\mathbf{N}}_i\vec{\mathbf{N}}_j = \vec{\mathbf{0}}$ for $i \neq j$,
5. $\vec{\mathbf{N}}_i^{m_i} = \vec{\mathbf{0}}$,
6. $A = \sum_{i=1}^k(\lambda_i\vec{\mathbf{Q}}_i + \vec{\mathbf{N}}_i)$.

The proof of **1** follows from clearing fractions in the partial fraction expansion of $1/p(\lambda)$:

$$1 = \sum_{i=1}^{k} a_i(\lambda)\frac{p(\lambda)}{(\lambda-\lambda_i)^{m_i}}.$$

The **projection property 2** follows by multiplication of identity **1** by $\vec{\mathbf{Q}}_i$ and then using **2**.

The proof of **3** starts by observing that $\vec{\mathbf{Q}}_i$ and $\vec{\mathbf{Q}}_j$ together contain all the factors of $p(A)$, therefore $\vec{\mathbf{Q}}_i\vec{\mathbf{Q}}_j = q(A)p(A)$ for some polynomial q. The Cayley-Hamilton theorem $p(A) = \vec{\mathbf{0}}$ finishes the proof.

To prove **4**, write $\vec{\mathbf{N}}_i\vec{\mathbf{N}}_j = (A-\lambda_i I)(A-\lambda_j I)\vec{\mathbf{Q}}_i\vec{\mathbf{Q}}_j$ and apply **3**.

To prove **5**, use $\vec{\mathbf{Q}}_i^{m_i} = \vec{\mathbf{Q}}_i$ (from **2**) to write $\vec{\mathbf{N}}_i^{m_i} = (A-\lambda_i I)^{m_i}\vec{\mathbf{Q}}_i = p(A) = \vec{\mathbf{0}}$.

To prove **6**, multiply **1** by A and rearrange as follows:

$$\begin{array}{rcl} A &=& \sum_{i=1}^k A\vec{\mathbf{Q}}_i \\ &=& \sum_{i=1}^k \lambda_i\vec{\mathbf{Q}}_i + (A-\lambda_i I)\vec{\mathbf{Q}}_i \\ &=& \sum_{i=1}^k \lambda_i\vec{\mathbf{Q}}_i + \vec{\mathbf{N}}_i \end{array}$$

To prove (2), multiply **1** by e^{At} and compute as follows:

$$\begin{aligned} e^{At} &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{At} \\ &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{\lambda_i It + (A-\lambda_i I)t} \\ &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{\lambda_i t} e^{(A-\lambda_i I)t} \\ &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{\lambda_i t} e^{\vec{\mathbf{Q}}_i(A-\lambda_i I)t} \\ &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{\lambda_i t} e^{\vec{\mathbf{N}}_i t} \\ &= \textstyle\sum_{i=1}^k \vec{\mathbf{Q}}_i e^{\lambda_i t} \sum_{j=0}^{m_1-1}(A-\lambda_i I)^j \,\frac{t^j}{j!} \end{aligned}$$

Solving Planar Systems $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$

A 2×2 real system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ can be solved in terms of the roots of the characteristic equation $\det(A - \lambda I) = 0$ and the real matrix A.

Theorem 20 (Planar System, Putzer's Spectral Formula)
Consider the real planar system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\vec{\mathbf{x}}(t)$ is given by the formula

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$$

where the 2×2 exponential matrix e^{At} is given as follows.

Real $\lambda_1 \neq \lambda_2$	$e^{At} = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I).$
Real $\lambda_1 = \lambda_2$	$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I).$
Complex $\lambda_1 = \overline{\lambda_2}$, $\lambda_1 = a + bi$, $b > 0$	$e^{At} = e^{at}\cos bt\, I + \dfrac{e^{at}\sin(bt)}{b}(A - aI).$

Proof: The formulas are from Putzer's algorithm, or equivalently, from the spectral formulas, with rearranged terms. The complex case is formally the real part of the distinct root case when $\lambda_2 = \overline{\lambda_1}$. The **spectral formula** is the analog of the second order equation formulas, Theorem 1 in Chapter 5.

Illustrations. Typical cases are represented by the following 2×2 matrices A, which correspond to roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ which are real distinct, real double or complex conjugate. The solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ is given here in two forms, by writing e^{At} using **1** a **spectral formula** and **2** Putzer's **spectral formula**.

$\lambda_1 = 5$, $\lambda_2 = 2$

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$$

Real distinct roots.

1 $$e^{At} = \frac{e^{5t}}{3}\begin{pmatrix} -3 & 3 \\ -6 & 6 \end{pmatrix} + \frac{e^{2t}}{-3}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

2 $$e^{At} = e^{5t} I + \frac{e^{2t} - e^{5t}}{2 - 5}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 3$

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

Real double root.

1 $$e^{At} = e^{3t}\left(I + t\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right)$$

2 $$e^{At} = e^{3t} I + t e^{3t}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$\lambda_1 = \overline{\lambda}_2 = 2 + 3i$

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

Complex conjugate roots.

1 $$e^{At} = 2\,\mathcal{R}\mathrm{e}\left(\frac{e^{2t+3it}}{2(3i)} \begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \right)$$

2 $$e^{At} = e^{2t}\cos 3tI + \frac{e^{2t}\sin 3t}{3} \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

The complex example is typical for real $n \times n$ matrices A with a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2$. Then $\vec{\mathbf{Q}}_2 = \overline{\vec{\mathbf{Q}}_1}$. The result is that λ_2 is not used and we write instead a simpler expression using the college algebra equality $z + \overline{z} = 2\,\mathcal{R}\mathrm{e}(z)$:

$$e^{\lambda_1 t}\vec{\mathbf{Q}}_1 + e^{\lambda_2 t}\vec{\mathbf{Q}}_2 = 2\,\mathcal{R}\mathrm{e}\left(e^{\lambda_1 t}\vec{\mathbf{Q}}_1\right).$$

This observation explains why e^{At} is real when A is real, by pairing complex conjugate eigenvalues in the spectral formula.