11.4 Matrix Exponential

The problem

$$\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t), \quad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$$

has a unique solution, according to the Picard-Lindelöf theorem. Solve the problem n times, when $\vec{\mathbf{x}}_0$ equals a column of the identity matrix, and write $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$ for the n solutions so obtained. Define the **matrix exponential** e^{At} by packaging these n solutions into a matrix:

$$e^{At} \equiv \langle \vec{\mathbf{w}}_1(t)|\ldots|\vec{\mathbf{w}}_n(t)\rangle.$$

By construction, any possible solution of $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$ can be uniquely expressed in terms of the matrix exponential e^{At} by the formula

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0).$$

Matrix Exponential Identities

Announced here and proved below are various formulas and identities for the matrix exponential e^{At}:

$\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$	Columns satisfy $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.
$e^{\vec{\mathbf{0}}} = I$	Where $\vec{\mathbf{0}}$ is the zero matrix.
$Be^{At} = e^{At}B$	If $AB = BA$.
$e^{At}e^{Bt} = e^{(A+B)t}$	If $AB = BA$.
$e^{At}e^{As} = e^{A(t+s)}$	Since At and As commute.
$\left(e^{At}\right)^{-1} = e^{-At}$	Equivalently, $e^{At}e^{-At} = I$.
$e^{At} = r_1(t)P_1 + \cdots + r_n(t)P_n$	Putzer's spectral formula — see page 816.
$e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 \neq \lambda_2$ real.
$e^{At} = e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 = \lambda_2$ real.
$e^{At} = e^{at}\cos bt\, I + \frac{e^{at}\sin bt}{b}(A - aI)$	A is 2×2, $\lambda_1 = \overline{\lambda}_2 = a + ib$, $b > 0$.
$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$	Picard series. See page 818.
$e^{At} = P^{-1}e^{Jt}P$	Jordan form $J = PAP^{-1}$.

Putzer's Spectral Formula

The spectral formula of Putzer applies to a system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ to find its general solution. The method uses matrices $P_1, \ldots, P_n$ constructed from A and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, matrix multiplication, and the solution $\vec{\mathbf{r}}(t)$ of the first order $n \times n$ initial value problem

$$\vec{\mathbf{r}}'(t) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & \lambda_n \end{pmatrix} \vec{\mathbf{r}}(t), \quad \vec{\mathbf{r}}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The system is solved by first order scalar methods and back-substitution. We will derive the formula separately for the 2×2 case (the one used most often) and the $n \times n$ case.

Spectral Formula 2×2

The general solution of the 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is given by the formula

$$\vec{\mathbf{x}}(t) = (r_1(t)P_1 + r_2(t)P_2)\,\vec{\mathbf{x}}(0),$$

where r_1, r_2, P_1, P_2 are defined as follows.

The eigenvalues $r = \lambda_1, \lambda_2$ are the two roots of the quadratic equation

$$\det(A - rI) = 0.$$

Define 2×2 matrices P_1, P_2 by the formulas

$$P_1 = I, \quad P_2 = A - \lambda_1 I.$$

The functions $r_1(t)$, $r_2(t)$ are defined by the differential system

$$\begin{cases} r_1' &= \lambda_1 r_1, & r_1(0) = 1, \\ r_2' &= \lambda_2 r_2 + r_1, & r_2(0) = 0. \end{cases}$$

Proof: The Cayley-Hamilton formula $(A - \lambda_1 I)(A - \lambda_2 I) = \vec{\mathbf{0}}$ is valid for any 2×2 matrix A and the two roots $r = \lambda_1, \lambda_2$ of the determinant equality $\det(A - rI) = 0$. The Cayley-Hamilton formula is the same as $(A - \lambda_2)P_2 = \vec{\mathbf{0}}$, which implies the identity $AP_2 = \lambda_2 P_2$. Compute as follows.

$$\begin{aligned} \vec{\mathbf{x}}'(t) &= (r_1'(t)P_1 + r_2'(t)P_2)\,\vec{\mathbf{x}}(0) \\ &= (\lambda_1 r_1(t)P_1 + r_1(t)P_2 + \lambda_2 r_2(t)P_2)\,\vec{\mathbf{x}}(0) \\ &= (r_1(t)A + \lambda_2 r_2(t)P_2)\,\vec{\mathbf{x}}(0) \\ &= (r_1(t)A + r_2(t)AP_2)\,\vec{\mathbf{x}}(0) \\ &= A\,(r_1(t)I + r_2(t)P_2)\,\vec{\mathbf{x}}(0) \\ &= A\vec{\mathbf{x}}(t). \end{aligned}$$

This proves that $\vec{\mathbf{x}}(t)$ is a solution. Because $\Phi(t) \equiv r_1(t)P_1 + r_2(t)P_2$ satisfies $\Phi(0) = I$, then any possible solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ can be represented by the given formula. The proof is complete.

Real Distinct Eigenvalues. Suppose A is 2×2 having real distinct eigenvalues λ_1, λ_2 and $\vec{\mathbf{x}}(0)$ is real. Then

$$r_1 = e^{\lambda_1 t}, \quad r_2 = \frac{e^{\lambda_1 t} - e^{\lambda_2 T}}{\lambda_1 - \lambda_2}$$

and

$$\vec{\mathbf{x}}(t) = \left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I) \right) \vec{\mathbf{x}}(0).$$

The matrix exponential formula for real distinct eigenvalues:

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

Real Equal Eigenvalues. Suppose A is 2×2 having real equal eigenvalues $\lambda_1 = \lambda_2$ and $\vec{\mathbf{x}}(0)$ is real. Then $r_1 = e^{\lambda_1 t}$, $r_2 = te^{\lambda_1 t}$ and

$$\vec{\mathbf{x}}(t) = \left(e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I) \right) \vec{\mathbf{x}}(0).$$

The matrix exponential formula for real equal eigenvalues:

$$e^{At} = e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I).$$

Complex Eigenvalues. Suppose A is 2×2 having complex eigenvalues $\lambda_1 = a + bi$ with $b > 0$ and $\lambda_2 = a - bi$. If $\vec{\mathbf{x}}(0)$ is real, then a real solution is obtained by taking the real part of the spectral formula. This formula is formally identical to the case of real distinct eigenvalues. Then

$$\begin{aligned}
\mathcal{R}\mathrm{e}(\vec{\mathbf{x}}(t)) &= \left(\mathcal{R}\mathrm{e}(r_1(t))I + \mathcal{R}\mathrm{e}(r_2(t)(A - \lambda_1 I))\right) \vec{\mathbf{x}}(0) \\
&= \left(\mathcal{R}\mathrm{e}(e^{(a+ib)t})I + \mathcal{R}\mathrm{e}(e^{at}\frac{\sin bt}{b}(A - (a + ib)I)) \right) \vec{\mathbf{x}}(0) \\
&= \left(e^{at}\cos bt\, I + e^{at}\frac{\sin bt}{b}(A - aI)) \right) \vec{\mathbf{x}}(0)
\end{aligned}$$

The matrix exponential formula for complex conjugate eigenvalues:

$$e^{At} = e^{at}\left(\cos bt\, I + \frac{\sin bt}{b}(A - aI)) \right).$$

How to Remember Putzer's 2×2 Formula. The expressions

$$
e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I), \qquad (1)
$$
$$
r_1(t) = e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton quotient for r_1. It has limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$
r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}.
$$

Taking real parts of expression (1) gives the complex case formula.

Spectral Formula $n \times n$

The general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is given by the formula

$$
\vec{\mathbf{x}}(t) = (r_1(t)P_1 + r_2(t)P_2 + \cdots + r_n(t)P_n)\,\vec{\mathbf{x}}(0),
$$

where r_1, r_2, …, r_n, P_1, P_2, …, P_n are defined as follows.

The eigenvalues $r = \lambda_1, \ldots, \lambda_n$ are the roots of the polynomial equation

$$
\det(A - rI) = 0.
$$

Define $n \times n$ matrices P_1, …, P_n by the formulas

$$
P_1 = I, \quad P_k = P_{k-1}(A - \lambda_{k-1}I) = \Pi_{j=1}^{k-1}(A - \lambda_j I), \quad k = 2, \ldots, n.
$$

The functions $r_1(t)$, …, $r_n(t)$ are defined by the differential system

$$
\begin{array}{rcll}
r_1' & = & \lambda_1 r_1, & r_1(0) = 1, \\
r_2' & = & \lambda_2 r_2 + r_1, & r_2(0) = 0, \\
 & \vdots & & \\
r_n' & = & \lambda_n r_n + r_{n-1}, & r_n(0) = 0.
\end{array}
$$

Proof: The Cayley-Hamilton formula $(A - \lambda_1 I) \cdots (A - \lambda_n I) = \vec{\mathbf{0}}$ is valid for any $n \times n$ matrix A and the n roots $r = \lambda_1, \ldots, \lambda_n$ of the determinant equality $\det(A - rI) = 0$. Two facts will be used: (1) The Cayley-Hamilton formula implies $AP_n = \lambda_n P_n$; (2) The definition of P_k implies $\lambda_k P_k + P_{k+1} = AP_k$ for $1 \le k \le n-1$. Compute as follows.

$$
\boxed{1} \quad \vec{\mathbf{x}}'(t) = (r_1'(t)P_1 + \cdots + r_n'(t)P_n)\,\vec{\mathbf{x}}(0)
$$
$$
\boxed{2} \quad = \left(\sum_{k=1}^{n} \lambda_k r_k(t) P_k + \sum_{k=2}^{n} r_{k-1} P_k\right) \vec{\mathbf{x}}(0)
$$

$$\boxed{3} \quad = \left(\sum_{k=1}^{n-1} \lambda_k r_k(t) P_k + r_n(t)\lambda_n P_n + \sum_{k=1}^{n-1} r_k P_{k+1} \right) \vec{\mathbf{x}}(0)$$

$$\boxed{4} \quad = \left(\sum_{k=1}^{n-1} r_k(t)(\lambda_k P_k + P_{k+1}) + r_n(t)\lambda_n P_n \right) \vec{\mathbf{x}}(0)$$

$$\boxed{5} \quad = \left(\sum_{k=1}^{n-1} r_k(t) A P_k + r_n(t) A P_n \right) \vec{\mathbf{x}}(0)$$

$$\boxed{6} \quad = A \left(\sum_{k=1}^{n} r_k(t) P_k \right) \vec{\mathbf{x}}(0)$$

$$\boxed{7} \quad = A\vec{\mathbf{x}}(t).$$

Details: $\boxed{1}$ Differentiate the formula for $\vec{\mathbf{x}}(t)$. $\boxed{2}$ Use the differential equations for $r_1,\ldots,r_n$. $\boxed{3}$ Split off the last term from the first sum, then re-index the last sum. $\boxed{4}$ Combine the two sums. $\boxed{5}$ Use the recursion for P_k and the Cayley-Hamilton formula $(A - \lambda_n I)P_n = \vec{\mathbf{0}}$. $\boxed{6}$ Factor out A on the left. $\boxed{7}$ Apply the definition of $\vec{\mathbf{x}}(t)$.

This proves that $\vec{\mathbf{x}}(t)$ is a solution. Because $\Phi(t) \equiv \sum_{k=1}^{n} r_k(t)P_k$ satisfies $\Phi(0) = I$, then any possible solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ can be so represented. The proof is complete.

Proofs of Matrix Exponential Properties

Verify $\left(e^{At}\right)' = Ae^{At}$. Let $\vec{\mathbf{x}}_0$ denote a column of the identity matrix. Define $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$. Then

$$\begin{aligned} \left(e^{At}\right)' \vec{\mathbf{x}}_0 &= \vec{\mathbf{x}}'(t) \\ &= A\vec{\mathbf{x}}(t) \\ &= Ae^{At}\vec{\mathbf{x}}_0. \end{aligned}$$

Because this identity holds for all columns of the identity matrix, then $(e^{At})'$ and Ae^{At} have identical columns, hence we have proved the identity $\left(e^{At}\right)' = Ae^{At}$.

Verify $AB = BA$ implies $Be^{At} = e^{At}B$. Define $\vec{\mathbf{w}}_1(t) = e^{At}B\vec{\mathbf{w}}_0$ and $\vec{\mathbf{w}}_2(t) = Be^{At}\vec{\mathbf{w}}_0$. Calculate $\vec{\mathbf{w}}_1'(t) = A\vec{\mathbf{w}}_1(t)$ and $\vec{\mathbf{w}}_2'(t) = BAe^{At}\vec{\mathbf{w}}_0 = ABe^{At}\vec{\mathbf{w}}_0 = A\vec{\mathbf{w}}_2(t)$, due to $BA = AB$. Because $\vec{\mathbf{w}}_1(0) = \vec{\mathbf{w}}_2(0) = \vec{\mathbf{w}}_0$, then the uniqueness assertion of the Picard-Lindelöf theorem implies that $\vec{\mathbf{w}}_1(t) = \vec{\mathbf{w}}_2(t)$. Because $\vec{\mathbf{w}}_0$ is any vector, then $e^{At}B = Be^{At}$. The proof is complete.

Verify $e^{At}e^{Bt} = e^{(A+B)t}$. Let $\vec{\mathbf{x}}_0$ be a column of the identity matrix. Define $\vec{\mathbf{x}}(t) = e^{At}e^{Bt}\vec{\mathbf{x}}_0$ and $\vec{\mathbf{y}}(t) = e^{(A+B)t}\vec{\mathbf{x}}_0$. We must show that $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$ for all t. Define $\vec{\mathbf{u}}(t) = e^{Bt}\vec{\mathbf{x}}_0$. We will apply the result $e^{At}B = Be^{At}$, valid for $BA = AB$. The details:

$$\begin{aligned} \vec{\mathbf{x}}'(t) &= \left(e^{At}\vec{\mathbf{u}}(t)\right)' \\ &= Ae^{At}\vec{\mathbf{u}}(t) + e^{At}\vec{\mathbf{u}}'(t) \\ &= A\vec{\mathbf{x}}(t) + e^{At}B\vec{\mathbf{u}}(t) \\ &= A\vec{\mathbf{x}}(t) + Be^{At}\vec{\mathbf{u}}(t) \\ &= (A+B)\vec{\mathbf{x}}(t). \end{aligned}$$

We also know that $\vec{\mathbf{y}}'(t) = (A+B)\vec{\mathbf{y}}(t)$ and since $\vec{\mathbf{x}}(0) = \vec{\mathbf{y}}(0) = \vec{\mathbf{x}}_0$, then the Picard-Lindelöf theorem implies that $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$ for all t. This completes the proof.

Verify $e^{At}e^{As} = e^{A(t+s)}$. Let t be a variable and consider s fixed. Define $\vec{\mathbf{x}}(t) = e^{At}e^{As}\vec{\mathbf{x}}_0$ and $\vec{\mathbf{y}}(t) = e^{A(t+s)}\vec{\mathbf{x}}_0$. Then $\vec{\mathbf{x}}(0) = \vec{\mathbf{y}}(0)$ and both satisfy the differential equation $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$. By the uniqueness in the Picard-Lindelöf theorem, $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$, which implies $e^{At}e^{As} = e^{A(t+s)}$. The proof is complete.

Verify $e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$. The idea of the proof is to apply Picard iteration. By definition, the columns of e^{At} are vector solutions $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$ whose values at $t=0$ are the corresponding columns of the $n \times n$ identity matrix. According to the theory of Picard iterates, a particular iterate is defined by

$$\vec{\mathbf{y}}_{n+1}(t) = \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_n(r)dr, \quad n \geq 0.$$

The vector $\vec{\mathbf{y}}_0$ equals some column of the identity matrix. The Picard iterates can be found explicitly, as follows.

$$\begin{aligned}
\vec{\mathbf{y}}_1(t) &= \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_0 dr \\
&= (I + At)\vec{\mathbf{y}}_0, \\
\vec{\mathbf{y}}_2(t) &= \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_1(r)dr \\
&= \vec{\mathbf{y}}_0 + \int_0^t A(I + At)\vec{\mathbf{y}}_0 dr \\
&= \left(I + At + A^2t^2/2\right)\vec{\mathbf{y}}_0, \\
&\vdots \\
\vec{\mathbf{y}}_n(t) &= \left(I + At + A^2\tfrac{t^2}{2} + \cdots + A^n\tfrac{t^n}{n!}\right)\vec{\mathbf{y}}_0.
\end{aligned}$$

The Picard-Lindelöf theorem implies that for $\vec{\mathbf{y}}_0$ = column k of the identity matrix,

$$\lim_{n\to\infty} \vec{\mathbf{y}}_n(t) = \vec{\mathbf{w}}_k(t).$$

This being valid for each index k, then the columns of the matrix sum

$$\sum_{m=0}^{N} A^m \frac{t^m}{m!}$$

converge as $N \to \infty$ to $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$. This implies the matrix identity

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}.$$

The proof is complete.

Computing e^{At}

Theorem 13 (Computing e^{Jt} for J Triangular)
If J is an upper triangular matrix, then a column $\vec{\mathbf{u}}(t)$ of e^{Jt} can be computed by solving the system $\vec{\mathbf{u}}'(t) = J\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{v}}$, where $\vec{\mathbf{v}}$ is the

corresponding column of the identity matrix. This problem can always be solved by first-order scalar methods of growth-decay theory and the integrating factor method.

Theorem 14 (Exponential of a Diagonal Matrix)
For real or complex constants $\lambda_1, \ldots, \lambda_n$,

$$e^{\mathbf{diag}(\lambda_1,\ldots,\lambda_n)t} = \mathbf{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right).$$

Theorem 15 (Block Diagonal Matrix)
If $A = \mathbf{diag}(B_1, \ldots, B_k)$ and each of $B_1, \ldots, B_k$ is a square matrix, then

$$e^{At} = \mathbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right).$$

Theorem 16 (Complex Exponential)
Given real a, b, then

$$e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t} = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}.$$

Exercises 11.4

Matrix Exponential.

1. (Picard) Let A be real 2×2. Write out the two initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$ of e^{At}.

2. (Picard) Let A be real 3×3. Write out the three initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$, $\vec{\mathbf{w}}_3(t)$ of e^{At}.

3. (Definition) Let A be real 2×2. Show that the solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0$.

4. Definition Let A be real $n \times n$. Show that the solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Matrix Exponential 2×2. Find e^{At} using the formula $e^{At} = \langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \rangle$ and the corresponding systems $\vec{\mathbf{w}}_1' = A\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{w}}_2' = A\vec{\mathbf{w}}_2$, $\vec{\mathbf{w}}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In these exercises A is triangular so that first-order methods can solve the systems.

5. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

6. $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

8. $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

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