11.2 Basic First-order System Methods

Solving 2×2 Systems

It is shown here that *any* constant linear system

$$\vec{\mathbf{u}}' = A\vec{\mathbf{u}}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be solved by one of the following elementary methods.

(a) The integrating factor method for $y' = p(x)y + q(x)$.

(b) The second order constant coefficient formulas in Theorem 1, Chapter 5.

Triangular A. Let's assume $b = 0$, so that A is lower triangular. The upper triangular case is handled similarly. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ has the scalar form

$$\begin{aligned} u_1' &= au_1, \\ u_2' &= cu_1 + du_2. \end{aligned}$$

The first differential equation is solved by the growth/decay formula:

$$u_1(t) = u_0 e^{at}.$$

Then substitute the answer just found into the second differential equation to give

$$u_2' = du_2 + cu_0 e^{at}.$$

This is a linear first order equation of the form $y' = p(x)y + q(x)$, to be solved by the integrating factor method. Therefore, a triangular system can always be solved by the first order integrating factor method.

An illustration. Let us solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ for the triangular matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \text{representing} \quad \begin{cases} u_1' = u_1, \\ u_2' = 2u_1 + u_2. \end{cases}$$

The first equation $u_1' = u_1$ has solution $u_1 = c_1 e^t$. The second equation $u_2' = 2u_1 + u_2$ becomes upon substitution of $u_1 = c_1 e^t$ the new equation

$$u_2' = 2c_1 e^t + u_2,$$

which is a first order linear differential equation with linear integrating factor method solution $u_2 = (2c_1 t + c_2)e^t$. The general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ in scalar form is

$$u_1 = c_1 e^t, \quad u_2 = 2c_1 te^t + c_2 e^t.$$

The **vector form** of the general solution is

$$\vec{\mathbf{u}}(t) = c_1 \begin{pmatrix} e^t \\ 2te^t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

The **vector basis** is the set

$$\mathcal{B} = \left\{ \begin{pmatrix} e^t \\ 2te^t \end{pmatrix}, \begin{pmatrix} 0 \\ e^t \end{pmatrix} \right\}.$$

Non-Triangular A. In order that A be non-triangular, both $b \neq 0$ and $c \neq 0$ must be satisfied. The scalar form of the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is

$$\begin{aligned} u_1' &= au_1 + bu_2, \\ u_2' &= cu_1 + du_2. \end{aligned}$$

Theorem 1 (Solving Non-Triangular $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$)
Solutions u_1, u_2 of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ are linear combinations of the list of Euler solution atoms obtained from the roots r of the quadratic equation

$$\det(A - rI) = 0.$$

Proof: The method: differentiate the first equation, then use the equations to eliminate u_2, u_2'. The result is a second order differential equation for u_1. The same differential equation is satisfied also for u_2. The details:

$u_1'' = au_1' + bu_2'$	Differentiate the first equation.
$\quad = au_1' + bcu_1 + bdu_2$	Use equation $u_2' = cu_1 + du_2$.
$\quad = au_1' + bcu_1 + d(u_1' - au_1)$	Use equation $u_1' = au_1 + bu_2$.
$\quad = (a+d)u_1' + (bc - ad)u_1$	Second order equation for u_1 found

The characteristic equation of $u_1'' - (a+d)u_1' + (ad - bc)u_1 = 0$ is

$$r^2 - (a+d)r + (bc - ad) = 0.$$

Finally, we show the expansion of $\det(A - rI)$ is the same characteristic polynomial:

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix} \\ &= (a-r)(d-r) - bc \\ &= r^2 - (a+d)r + ad - bc. \end{aligned}$$

The proof is complete.

The reader can verify that the differential equation for u_1 or u_2 is exactly

$$u'' - \mathbf{trace}(A)u' + \det(A)u = 0.$$

Assume below that A is non-triangular, meaning $b \neq 0$ and $c \neq 0$.

Finding u_1. Apply the second order formulas, Theorem 1 in Chapter 5, to solve for u_1. This involves writing a list of Euler solution atoms corresponding to the two roots of the characteristic equation $r^2 - (a + d)r + ad - bc = 0$, followed by expressing u_1 as a linear combination of the two Euler atoms.

Finding u_2. Isolate u_2 in the first differential equation by division:

$$u_2 = \frac{1}{b}(u_1' - au_1).$$

The two formulas for u_1, u_2 represent the general solution of the system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, when A is 2×2.

An illustration. Let's solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ when

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \text{representing} \quad \begin{cases} u_1' &= u_1 + 2u_2, \\ u_2' &= 2u_1 + u_2. \end{cases}$$

The equation $\det(A - rI) = 0$ is $(1-r)^2 - 4 = 0$ with roots $r = -1$ and $r = 3$. The Euler solution atom list is $L = \{e^{-t}, e^{3t}\}$. Then the linear combination of Euler atoms is $u_1 = c_1 e^{-t} + c_2 e^{3t}$. The first equation $u_1' = u_1 + 2u_2$ implies $u_2 = \frac{1}{2}(u_1' - u_1)$. The scalar general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is then

$$u_1 = c_1 e^{-t} + c_2 e^{3t}, \quad u_2 = -c_1 e^{-t} + c_2 e^{3t}.$$

In vector form, the general solution is

$$\vec{\mathbf{u}} = c_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}.$$

Triangular Methods

Diagonal $n \times n$ **matrix** $A = \textbf{diag}(a_1, \ldots, a_n)$. Then the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is a set of uncoupled scalar growth/decay equations:

$$\begin{aligned} x_1'(t) &= a_1 x_1(t), \\ x_2'(t) &= a_2 x_2(t), \\ &\vdots \\ x_n'(t) &= a_n x_n(t). \end{aligned}$$

The solution to the system is given by the formulas

$$\begin{aligned} x_1(t) &= c_1 e^{a_1 t}, \\ x_2(t) &= c_2 e^{a_2 t}, \\ &\vdots \\ x_n(t) &= c_n e^{a_n t}. \end{aligned}$$

The numbers $c_1, \ldots, c_n$ are arbitrary constants.

Triangular $n \times n$ matrix A. If a linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has a square triangular matrix A, then the system can be solved by first order scalar methods. To illustrate the ideas, consider the 3×3 linear system

$$\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 4 & 4 \end{pmatrix} \vec{\mathbf{x}}.$$

The coefficient matrix A is *lower triangular*. In scalar form, the system is given by the equations

$$\begin{aligned} x_1'(t) &= 2x_1(t), \\ x_2'(t) &= 3x_1(t) + 3x_2(t), \\ x_3'(t) &= 4x_1(t) + 4x_2(t) + 4x_3(t). \end{aligned}$$

A recursive method. The system is solved recursively by first order scalar methods only, starting with the first equation $x_1'(t) = 2x_1(t)$. This growth equation has general solution $x_1(t) = c_1 e^{2t}$. The second equation then becomes the first order linear equation

$$\begin{aligned} x_2'(t) &= 3x_1(t) + 3x_2(t) \\ &= 3x_2(t) + 3c_1 e^{2t}. \end{aligned}$$

The integrating factor method applies to find the general solution $x_2(t) = -3c_1 e^{2t} + c_2 e^{3t}$. The third and last equation becomes the first order linear equation

$$\begin{aligned} x_3'(t) &= 4x_1(t) + 4x_2(t) + 4x_3(t) \\ &= 4x_3(t) + 4c_1 e^{2t} + 4(-3c_1 e^{2t} + c_2 e^{3t}). \end{aligned}$$

The integrating factor method is repeated to find the general solution $x_3(t) = 4c_1 e^{2t} - 4c_2 e^{3t} + c_3 e^{4t}$.

In summary, the scalar general solution to the system is given by the formulas

$$\begin{aligned} x_1(t) &= c_1 e^{2t}, \\ x_2(t) &= -3c_1 e^{2t} + c_2 e^{3t}, \\ x_3(t) &= 4c_1 e^{2t} - 4c_2 e^{3t} + c_3 e^{4t}. \end{aligned}$$

Structure of solutions. A system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for $n \times n$ triangular A has component solutions $x_1(t), \ldots, x_n(t)$ given as polynomials times exponentials. The exponential factors $e^{a_{11}t}, \ldots, e^{a_{nn}t}$ are expressed in terms of the diagonal elements $a_{11}, \ldots, a_{nn}$ of the matrix A. Fewer than n distinct exponential factors may appear, due to duplicate diagonal elements. These duplications cause the polynomial factors to appear. The reader is invited to work out the solution to the system below, which has duplicate diagonal entries $a_{11} = a_{22} = a_{33} = 2$.

$$\begin{aligned} x_1'(t) &= 2x_1(t), \\ x_2'(t) &= 3x_1(t) + 2x_2(t), \\ x_3'(t) &= 4x_1(t) + 4x_2(t) + 2x_3(t). \end{aligned}$$

The solution, given below, has polynomial factors t and t^2, appearing because of the duplicate diagonal entries $2, 2, 2$, and only one exponential factor e^{2t}.

$$\begin{aligned} x_1(t) &= c_1 e^{2t}, \\ x_2(t) &= 3c_1 t e^{2t} + c_2 e^{2t}, \\ x_3(t) &= 4c_1 t e^{2t} + 6c_1 t^2 e^{2t} + 4c_2 t e^{2t} + c_3 e^{2t}. \end{aligned}$$

Conversion to Systems

Routinely converted to a system of equations of first order are scalar second order linear differential equations, systems of scalar second order linear differential equations and scalar linear differential equations of higher order.

Scalar second order linear equations. Consider an equation $au'' + bu' + cu = f$ where $a \neq 0$, b, c, f are allowed to depend on t, $' = d/dt$. Define the **position-velocity substitution**

$$x(t) = u(t), \quad y(t) = u'(t).$$

Then $x' = u' = y$ and $y' = u'' = (-bu' - cu + f)/a = -(b/a)y - (c/a)x + f/a$. The resulting system is equivalent to the second order equation, in the sense that the position-velocity substitution equates solutions of one system to the other:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{c(t)}{a(t)} x(t) - \frac{b(t)}{a(t)} y(t) + \frac{f(t)}{a(t)}. \end{aligned}$$

The case of constant coefficients and f a function of t arises often enough to isolate the result for further reference.

Theorem 2 (System Equivalent to Second Order Linear)
Let $a \neq 0$, b, c be constants and $f(t)$ continuous. Then $au''+bu'+cu = f(t)$ is equivalent to the first order system

$$a\vec{\mathbf{w}}\,'(t) = \begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix} \vec{\mathbf{w}}\,(t) + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \vec{\mathbf{w}}\,(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}.$$

Converting second order systems to first order systems. A similar position-velocity substitution can be carried out on a system of two second order linear differential equations. Assume

$$\begin{aligned} a_1 u_1'' + b_1 u_1' + c_1 u_1 &= f_1, \\ a_2 u_2'' + b_2 u_2' + c_2 u_2 &= f_2. \end{aligned}$$

Then the preceding methods for the scalar case give the equivalence

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}' = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -c_1 & -b_1 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -c_2 & -b_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix} + \begin{pmatrix} 0 \\ f_1 \\ 0 \\ f_2 \end{pmatrix}.$$

Coupled spring-mass systems. Springs connecting undamped coupled masses were considered at the beginning of this chapter, page 786. Typical equations are

$$(1)\qquad \begin{aligned} m_1x_1''(t) &= -k_1x_1(t) + k_2[x_2(t) - x_1(t)], \\ m_2x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\ m_3x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4x_3(t). \end{aligned}$$

The equations can be represented by a second order linear system of dimension 3 of the form $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}}$, where the **position** $\vec{\mathbf{x}}$, the **mass matrix** M and the **Hooke's matrix** K are given by the equalities

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$K = \begin{pmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & -k_3 & -(k_3+k_4) \end{pmatrix}.$$

Systems of second order linear equations. A second order system $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ is called a **forced system** and $\vec{\mathbf{F}}$ is called the **external vector force**. Such a system can always be converted to a second order system where the mass matrix is the identity, by multiplying by M^{-1}:

$$\vec{\mathbf{x}}'' = M^{-1}K\vec{\mathbf{x}} + M^{-1}\vec{\mathbf{F}}(t).$$

The benign form $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}} + \vec{\mathbf{G}}(t)$, where $A = M^{-1}K$ and $\vec{\mathbf{G}} = M^{-1}\vec{\mathbf{F}}$, admits a block matrix conversion into a first order system:

$$\frac{d}{dt}\begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} = \left(\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right) \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} + \begin{pmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{G}}(t) \end{pmatrix}.$$

Damped second order systems. The addition of a dashpot to each of the masses gives a **damped second order system** with forcing

$$M\vec{\mathbf{x}}'' = B\vec{\mathbf{x}}' + K\vec{\mathbf{X}} + \vec{\mathbf{F}}(t).$$

In the case of one scalar equation, the matrices M, B, K are constants m, $-c$, $-k$ and the external force is a scalar function $f(t)$, hence the system becomes the classical damped spring-mass equation

$$mx'' + cx' + kx = f(t).$$

A useful way to write the first order system is to introduce variable $\vec{\mathbf{u}} = M\vec{\mathbf{x}}$, in order to obtain the two equations

$$\vec{\mathbf{u}}' = M\vec{\mathbf{x}}', \quad \vec{\mathbf{u}}'' = B\vec{\mathbf{x}}' + K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t).$$

Then a first order system in block matrix form is given by

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array}\right) \frac{d}{dt} \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} = \left(\begin{array}{c|c} 0 & M \\ \hline K & B \end{array}\right) \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} + \begin{pmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{F}}(t) \end{pmatrix}.$$

The benign form $\vec{\mathbf{x}}'' = M^{-1}B\vec{\mathbf{x}}' + M^{-1}K\vec{\mathbf{x}} + M^{-1}\vec{\mathbf{F}}(t)$, obtained by left-multiplication by M^{-1}, can be similarly written as a first order system in block matrix form.

$$\frac{d}{dt} \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} = \left(\begin{array}{c|c} 0 & I \\ \hline M^{-1}K & M^{-1}B \end{array}\right) \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix} + \begin{pmatrix} \vec{\mathbf{0}} \\ M^{-1}\vec{\mathbf{F}}(t) \end{pmatrix}.$$

Higher order linear equations. Every homogeneous nth order constant-coefficient linear differential equation

$$y^{(n)} = p_0 y + \cdots + p_{n-1} y^{(n-1)}$$

can be converted to a linear homogeneous vector-matrix system

$$\frac{d}{dx} \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ p_0 & p_1 & p_2 & \cdots & p_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

This is a linear system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ where $\vec{\mathbf{u}}$ is the $n \times 1$ column vector consisting of y and its successive derivatives, while the $n \times n$ matrix A is the classical **companion matrix** of the characteristic polynomial

$$r^n = p_0 + p_1 r + p_2 r^2 + \cdots + p_{n-1} r^{n-1}.$$

To illustrate, the companion matrix for $r^4 = a + br + cr^2 + dr^3$ is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

The preceding companion matrix has the following block matrix form, which is representative of all companion matrices.

$$A = \left(\begin{array}{c|c} \vec{\mathbf{0}} & I \\ \hline a & b \quad c \quad d \end{array}\right).$$

Continuous coefficients. It is routinely observed that the methods above for conversion to a first order system apply equally as well to higher order linear differential equations with continuous coefficients. To illustrate, the fourth order linear equation $y^{iv} = a(x)y + b(x)y' + c(x)y'' + d(x)y'''$ has first order system form $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ where A is the companion matrix for the polynomial $r^4 = a(x) + b(x)r + c(x)r^2 + d(x)r^3$, x held fixed.

Forced higher order linear equations. All that has been said above applies equally to a forced linear equation like

$$y^{iv} = 2y + \sin(x)y' + \cos(x)y'' + x^2y''' + f(x).$$

It has a conversion to a first order nonhomogeneous linear system

$$\vec{\mathbf{u}}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & \sin x & \cos x & x^2 \end{pmatrix} \vec{\mathbf{u}} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{pmatrix}, \quad \vec{\mathbf{u}} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$