



Game Theory Through Linear Algebra

By Sean Hurst

Introduction

Game Theory is the mathematical study of possible choices that players can make in games in order to win. Psychologists call it the theory of social interaction because it attempts to consider the choices that one player can make against another in a competitive setting. Although game theory is focused around board games, cards, and other competitive games, it can also be applied to military strategy in war. This project will demonstrate how Linear Algebra matrix computations can be used as a powerful tool to solve Game Theory problems.

Key Terms

First, there are some key terms that are needed to fully understand Game Theory problems:

- **Player:** A person or object that competes with other persons or objects and has a specific set of choices they can make.
- **Game:** A problem or situation in which the choices made by one player can affect all players involved.
- **Risk:** How much the player can lose if they make a negative choice.
- **Reward:** How much the player can gain if they make a positive choice.
- **The Nash Equilibrium:** Attempts to predict what choices will be made if multiple players are making decisions simultaneously. Its main objective is to maximize a player's payoff while reducing an opponent's. This makes the resulting sum of player's choices equal to zero.
 - Mathematically, we have two players A and B, each with their own choices $A = n$ and $B = m$. We also have a function P, which can be calculated as $A \times B$. The resulting matrix is known as the "Payoff Matrix".
- **Zero Sum Game:** A type of Nash Equilibrium in which a player's risks or rewards are diametrically opposed to their opponent's. This is typically seen in two-player games.
- **Reduction by Dominance:** Check whether a row or column in the matrix is dominated by (i.e. larger than) another row or column in the matrix and remove the dominated row or column. Repeat this until there are no dominated rows or columns left.
 - Dominance is defined as a choice having either equal or even better rewards than the next best choice. It eliminates other choices in order to present the best choice to a player so they can gain the most amount of reward.
 - Always start with the first row. Each number in a row must be greater than (or less than) other numbers in opposing rows. Otherwise, both rows and columns dominate each other.

Examples

➤ **The Prisoner's Dilemma**

The Prisoner's Dilemma is a classic Game Theory problem and is most commonly used to introduce the ideas of it.

The Problem:

The police arrest two criminals, Rob and Bill, for selling drugs. The cases are open and shut, and both will go to jail for two years. They have never met, have no connections to each other, and consequently have no loyalties to each other. However, the police notice that they look like two criminals who had committed a bank robbery together two weeks earlier and want to press for a confession from the prisoners. They offer both a deal:

1. If both deny they committed the bank robbery, they will go to jail for two years for selling drugs, just like they were supposed to.
2. If Rob confesses to the bank robbery but Bill does not, then Rob will only go to jail for one year while Bill will go to jail for ten years.
3. Likewise, if Bill confesses to the bank robbery but Rob does not, then Bill will only go to jail for one year while Rob goes away for ten years.
4. If both confess to the bank robbery, then both Rob and Bill will go to jail for three years.

Which is the mostly likely choice that Rob and Bill will make?

The Math:

The deal made by the police can be laid out in a chart.

		Bill	
		Confess	Deny
Rob	Confess	3,3	1,10
	Deny	10,1	2,2

Each number represents the amount of years either Rob or Bill will go away for. The most rational choice to make in this case would be to deny the robbery since it has the least amount of risk. However, this game demonstrates that the most rational decision isn't always the choice that people will take. It may be more beneficial for Rob and Bill if they confess on the chance that the other does not because they would only get less time in jail.

Since both Rob and Bill have two choices, confess or deny, we can calculate the payoff using the Nash Equilibrium. Thus,

$$n = m = 2$$

$$\text{Rob} = \text{Bill} = \{\text{Confess}, \text{Deny}\}$$

Representation: Confess = "0"; Deny = "1"

This produces two-dimensional vectors for each of the possible choices. Now we can calculate the cross product of each of the choices, or simply find the determinant of the matrix created since the matrix is two-dimensional.

If Rob confesses but Bill does not,

$$\text{Rob} \times \text{Bill} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

If Rob denies but Bill confesses,

$$\text{Rob} \times \text{Bill} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

If both Rob and Bill deny,

$$\text{Rob} \times \text{Bill} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0.$$

And finally, if both Rob and Bill confess,

$$\text{Rob} \times \text{Bill} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 0.$$

Here we can see the Nash Equilibrium come into play to decide which option is better. It is shown that if one prisoner confesses while the other denies, the resulting determinant does not equal zero. This makes sense because one prisoner only gets one year in jail while the other gets ten, and neither prisoner would want to deny and take this risk. This leads into the other two choices where both confess, or both deny. The determinant of these choices is zero, which means that these choices are more beneficial to the prisoners for getting less jail time.

Now the prisoners must decide if they trust each other. They have no contact with each other, so there is no way to coordinate a similar choice. It would be optimistic to say that both would deny they committed the bank robbery and get the regular two years. The prisoners take on a lot of risk if they were to deny since it would be the difference between two years and ten years in jail. Thus, to get the most reward, the most likely option to get the least amount of jail time would be to confess, since it is the difference between one year and three years in jail.

➤ **Rock, Paper, Scissors**

Rock, Paper, Scissors is a simple example of a Zero Sum Game. A Payoff Matrix can be determined the same way as the Prisoner's Dilemma's payoff, but this time produces a more symmetrical result. For example, we count the scores of two players over multiple games. For every win, a player adds one point, for every loss, a player subtracts a point, and for every tie neither adds nor subtracts a point from the score. The Payoff Matrix would look like this:

$$n = m = 3$$

$$P1 = P2 = \{\text{Rock, Paper, Scissors}\}$$

$$P1 \times P2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

This matrix is akin to a skew-symmetric matrix, which means that the game itself is symmetrical. If one player wins a point, the other player loses a point. If both players tie, then both players neither receive nor lose a point. This is the essence of a Zero Sum Game: for every positive choice a player makes toward a reward another player makes a negative choice that diametrically opposes it.

➤ **Football Strategies**

Different from the Nash Equilibrium and Zero Sum Games is the theory of Dominance, or more specifically Reduction by Dominance. This method uses dominant rows or columns in a Payoff Matrix to determine the highest reward out of different choices a player can make. This gets straight to the point, and most of the time is used to find absolute rewards in certain situations. For example, football teams use this method to decide which play to use against another team during a game to improve their chances of winning.

The Problem:

You are a coach for a football team that is currently playing offense. You want to break through the opposing team's defense and have five plays to choose from. However, you know the opposing team has three plays that they can choose from to try and stop you. Your team also has gathered information during the season on which plays gained the most yardage. Which strategy should you choose?

The Math:

The info the team has gathered is arranged in a five by three matrix:

$$\begin{bmatrix} 0 & -1 & 5 \\ 7 & 5 & 10 \\ 15 & -4 & -5 \\ 5 & 0 & 10 \\ -5 & -10 & 10 \end{bmatrix}.$$

It is arranged so that it can compare your team's five plays to the opposing team's three plays. We start by eliminating rows one, four, and five since the rewards are significantly larger no matter what the opposing team chooses for their strategy.

$$\begin{bmatrix} 7 & 5 & 10 \\ 15 & -4 & -5 \end{bmatrix}.$$

Now we move onto the columns. We now try to find the lowest reward for our team, but also try to guess which play the opposing team might choose since it will be our weakest play. We remove column one since it is dominated by column two and are left with a more refined offense against the opposing team.

$$\begin{bmatrix} 5 & 10 \\ -4 & -5 \end{bmatrix}.$$

Moving back to rows, we find that row one dominates row two and remove it from the matrix.

$$[5 \quad 10].$$

Finally, looking at columns again we remove column two since it is dominated by column one, and get our play for advancing past the opposing team.

$$[5].$$

Thus, our team should pick the second play to gain the most yardage against the opposing team.

Works Cited

Morgan, Julia. *Game Theory: Applications in Linear Algebra*. California State University, 2 Dec.

2013, web.csulb.edu/~jchang9/m247/m247_fa13_Julia_Lisa_Morgan_Chris_Lori.pdf.

Tripp, Erin. *Game Theory and Linear Algebra*. Syracuse University, 7 Aug. 2015,

etripp.expressions.syr.edu/wp-content/uploads/2015/08/finalpresentation.pdf.

“Prisoners' Dilemma and Nash Equilibrium.” *Khan Academy*, Khan Academy,

www.khanacademy.org/economics-finance-domain/ap-microeconomics/imperfect-competition/oligopoly-and-game-theory/v/prisoners-dilemma-and-nash-equilibrium.

Hazan, Elad. *Linear Programming and Game Theory*. Princeton University, 21 Apr. 2015,

www.cs.princeton.edu/courses/archive/spring15/cos511/notes/lec_11_kiran.pdf.

Ferguson, Thomas S. “Game Theory.” *Ucla.edu*, UCLA,

www.math.ucla.edu/~tom/Game_Theory/mat.pdf.

Wanar, Stefan. “Game Theory Subtopics.” *Summary: Game Theory*, Zweig Media, July 2007,

www.zweigmedia.com/RealWorld/Summary3b.html#rd.