Determining Temperature Distribution of a 2D Heated Plate

Kristen Stewart

Introduction to Conduction

Heat transfer by conduction, convection, and radiation are critical processes in mechanical engineering. This project will focus only on heat transfer by conduction through a two dimensional plate. On a molecular level, conduction is the process of transferring energy from higher energy particles to lower energy particles by collision of particles or lattice vibrations. On a macroscopic level, conduction is the heat transfer from a high temperature to a lower temperature of an object that is stationary. Conduction can occur through many mediums, including solids, liquids, and gases, as long as there is no bulk motion. Heat transfer by conduction is used in many daily applications, like heating a pan on a stove or the cooling of a room on a winter day by conduction through a window pane. The general form of the heat equation, which is the governing equation for all heat transfer applications, is expressed as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

This can be derived from energy balance principles (energy cannot be created nor destroyed) and applies to multidimensional heat transfer. However, assumptions can be made to certain cases which will make analysis much simpler.

1D Steady State Conduction

Assuming there is no time dependence for the temperature distribution and heat transfer only occurs in one dimension, there are a set of equations used to determine the heat rate transferred throughout the medium. For one dimensional heat transfer, the heat equation simplifies to

$$\frac{\partial^2 T}{\partial x^2}=0$$

This heat equation is important because it implies that the temperature distribution must be linear. Using a relationship known as Fourier's Law, which describes the heat flux through a plane wall that has a linear temperature distribution, the heat flux can be expressed as

$$q_x''=-k\frac{dT}{dx}$$

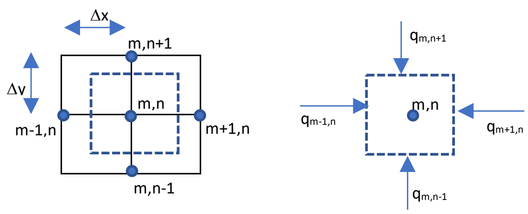
The heat flux, q_x'', is the heat rate per unit area, k is the thermal conductivity which is a property of the material, and $\frac{dT}{dx}$ is the temperature gradient in the direction of the heat transfer. Since there is a linear temperature distribution, $\frac{dT}{dx}=\frac{\Delta T}{L}$, where L is the length of the wall. The negative sign in Fourier's Law shows that the heat flux is in the direction of decreasing temperature, since heat is always transferred from a higher temperature to a lower temperature.

2D Steady State Conduction

Equations can be simplified to Fourier's Law for one-dimensional heat transfer, but the equations are more complex for heat transfer in multiple directions. Assuming the thermal conductivity, k, is a material constant, the heat equation for two dimensions can be simplified to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

This heat equation can be solved using analytical methods of partial differential equations, such as separation of variables. However, for engineering purposes, numerical methods can be used to closely estimate the solution of the heat equation. One of the most useful methods to solving this two dimensional heat equation is nodal analysis. In the case of a heated plate, the plate is broken up into discrete points, called nodes, that are evenly spaced throughout the plate. Using an energy balance equation at each node, the temperature of each node can be determined using Fourier's Law and a system of equations, as shown below. When converted into a matrix, linear algebra techniques can be used to solve the matrix equation.



$$E_{in(m,n)} = E_{out(m,n)}$$

$$q_{m-1,n} + q_{m+1,n} + q_{m,n-1} + q_{m,n+1} = 0$$

Heat flux is defined as heat rate per unit area, so heat rate can be rewritten as

$$Aq''_{m-1,n} + Aq''_{m+1,n} + Aq''_{m,n-1} + Aq''_{m,n+1} = 0.$$

Substituting in Fourier's Law for each heat flux,

$$A\left(-k\frac{T_{m,n} - T_{m-1,n}}{\Delta x}\right) + A\left(-k\frac{T_{m,n} - T_{m+1,n}}{\Delta x}\right) + A\left(-k\frac{T_{m,n} - T_{m,n-1}}{\Delta y}\right) + A\left(-k\frac{T_{m,n} - T_{m,n+1}}{\Delta y}\right) = 0$$

Since each node is equally spaced, Δx =Δy,

$$-\frac{Ak}{\Delta x}\Big(\left(T_{m,n} - T_{m-1,n}\right) + \left(T_{m,n} - T_{m+1,n}\right) + \left(T_{m,n} - T_{m,n-1}\right) + \left(T_{m,n} - T_{m,n+1}\right)\Big) = 0$$

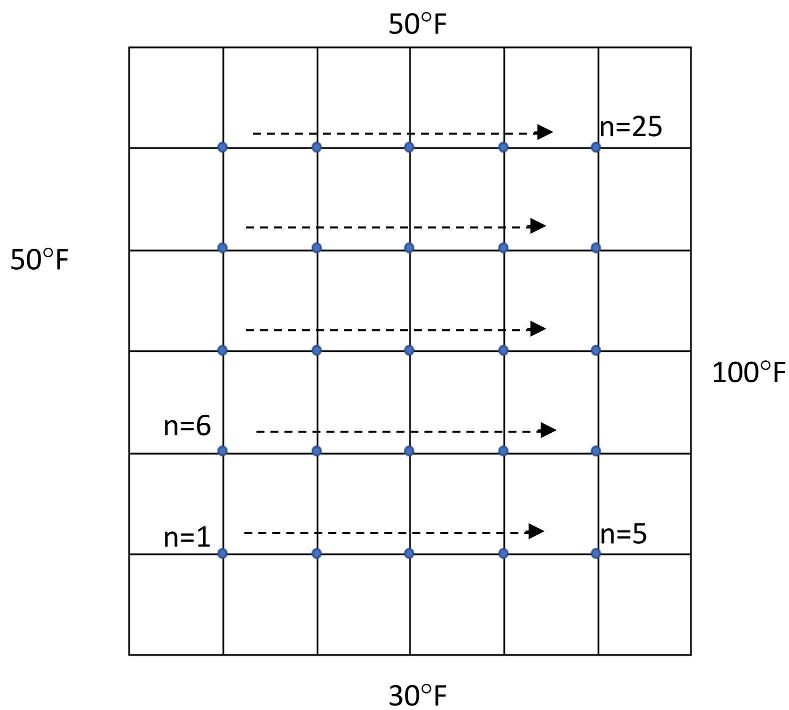
Rearranging,

$$4T_{m,n} - T_{m,n+1} - T_{m,n-1} - T_{m+1,n} - T_{m-1,n} = 0$$

Impact of Number of Nodes on Temperature Distribution

To illustrate the method of nodal analysis, there will be two heated plates, both with the same boundary conditions, but with a different number of nodes for each plate.

(1) n = 5:



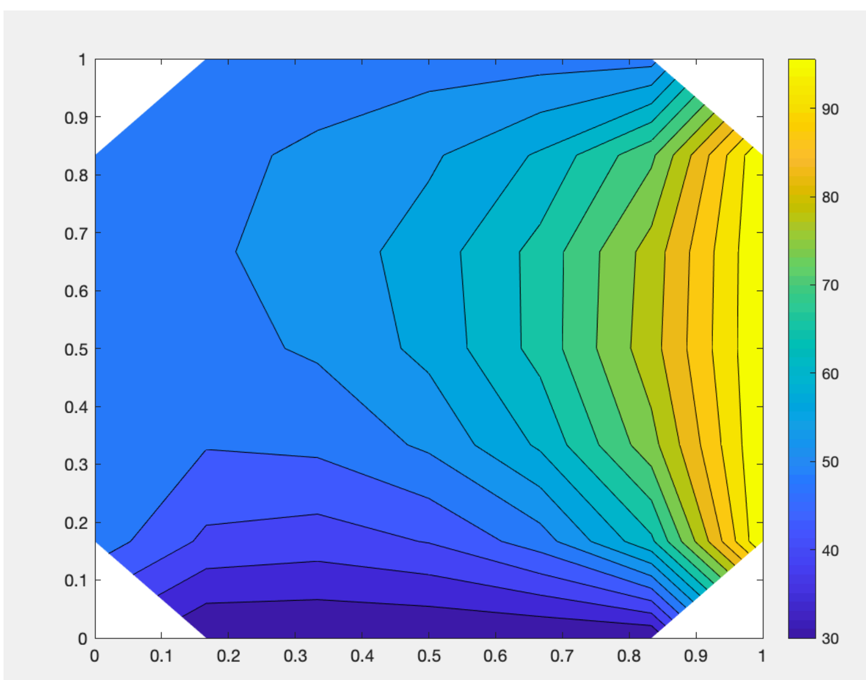
Using the equations derived in the previous section, the augmented matrix, [A|b], of temperatures at each node, starting from the bottom left node, is shown below.

4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	80
-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30
0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30
0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	30
0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	130
-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	100
0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	50
0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	100
0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	50
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	150

Since [A] is a square matrix and has a nonzero determinant, [A] has an inverse. Therefore, the solution to the matrix equation [A]**x** = **b**, can be expressed as **x** = $[A]^{-1}$**b**. The solution vector, **x,** is shown below.

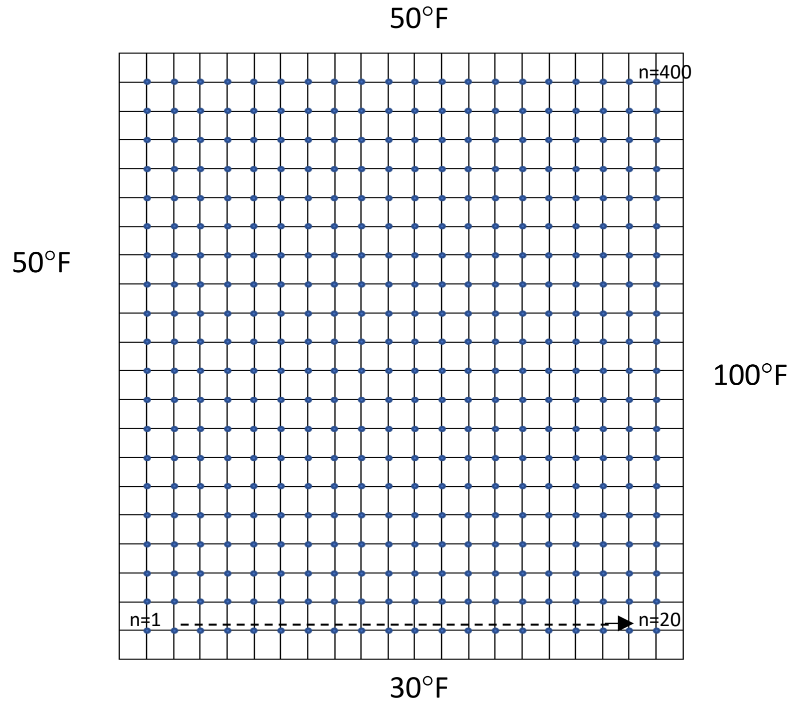
$$\mathbf{x} = \begin{bmatrix} T\_1 \\ T\_2 \\ T\_3 \\ T\_4 \\ T\_5 \\ T\_6 \\ T\_7 \\ T\_8 \\ T\_9 \\ T\_10 \\ T\_11 \\ T\_12 \\ T\_13 \\ T\_14 \\ T\_15 \\ T\_16 \\ T\_17 \\ T\_18 \\ T\_19 \\ T\_20 \\ T\_21 \\ T\_22 \\ T\_23 \\ T\_24 \\ T\_25 \end{bmatrix} = \begin{bmatrix} 42.1919192 \\ 41.0047591 \\ 43.3422688 \\ 49.6916278 \\ 64.0606061 \\ 47.7629176 \\ 48.4848485 \\ 52.6726884 \\ 61.3636364 \\ 76.5507964 \\ 50.3749029 \\ 52.4990287 \\ 57.5 \\ 66.5394328 \\ 80.7789433 \\ 51.2376651 \\ 53.6363636 \\ 58.28885 \\ 66.5151515 \\ 80.0255439 \\ 50.9393939 \\ 52.5199106 \\ 55.503885 \\ 61.2067793 \\ 72.8080808 \end{bmatrix}$$

Using this solution vector, a contour plot of the temperatures at each node on the plate was created and shown below, as a visual representation of the temperature distribution across the plate.

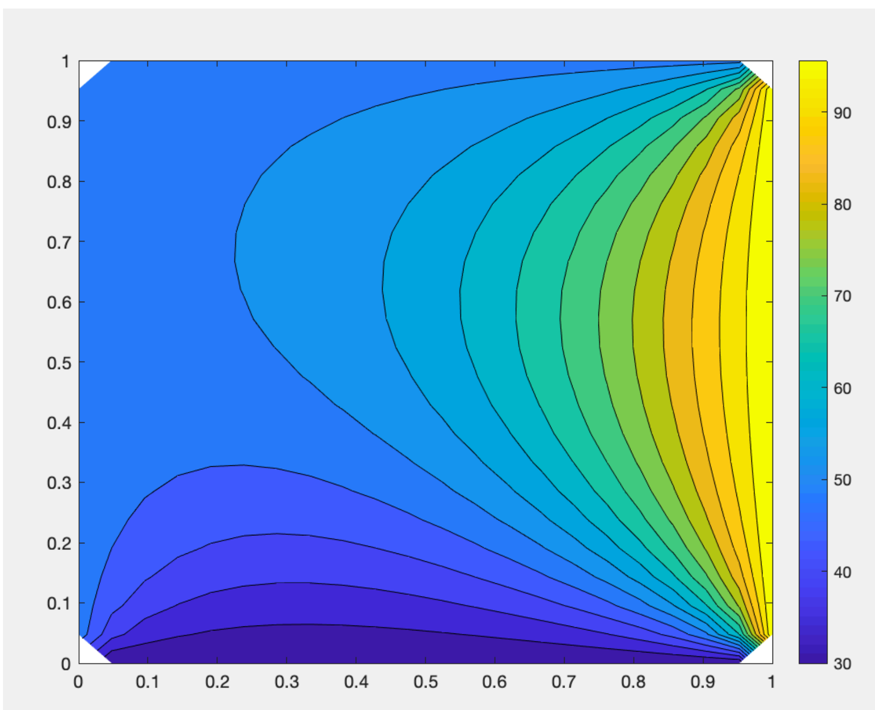


Now, the number of nodes along each row was increased from n = 5 to n=20 to show how the temperature distribution changes, with the same boundary conditions

(2) n = 20:



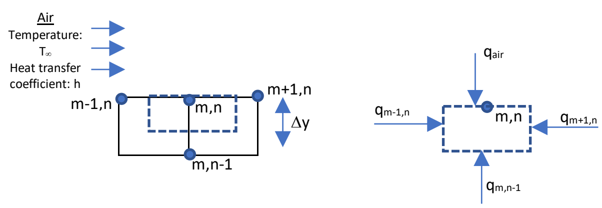
Using the same process as above, the matrix [A] of equations at each node can be obtained. The matrix is size 400x400, so the details are omitted. Once the vector of temperatures at each node is solved for, a contour plot of the temperature distribution can be created again.



For the same boundary conditions on a heated plate, as the number of nodes increases, the discontinuities in the temperature distribution across nodes transition to a more continuous distribution. To achieve the most accurate approximation, increase the number of nodes until there are an infinite number of nodes within the plate.

<u>Impact of Boundary Conditions on Temperature Distribution</u>

When the temperatures at the boundary of the heated plate are fixed, the temperature distribution will be similar to the previous examples. However, in engineering purposes, the boundary does not always have a prescribed temperature. Some important applications require convection on the outer surface, like air blowing over the surface, so heat is transferred from the air to the surface of the plate. The boundary conditions change, so the temperature distribution will be described below.



Using the same principles of energy balance and heat flux, the equation for the temperature at node (m,n) can be derived.

$$E_{in(m,n)} = E_{out(m,n)}$$

$$q_{m-1,n} + q_{m+1,n} + q_{m,n-1} + q_{air} = 0$$

Heat flux is defined as heat rate per unit area, so heat rate can be rewritten as

$$\frac{A}{2} q''_{m-1,n} + \frac{A}{2} q''_{m+1,n} + A q''_{m,n-1} + A q''_{air} = 0.$$

Substituting in Fourier's Law for each heat flux from the nodes and the relationship that $q''_{air} = h\left(T_\infty - T_{m,n}\right)$,

$$\frac{A}{2}\left(-k\frac{T_{m,n} - T_{m-1,n}}{\Delta x}\right) + \frac{A}{2}\left(-k\frac{T_{m,n} - T_{m+1,n}}{\Delta x}\right) + A\left(-k\frac{T_{m,n} - T_{m,n-1}}{\Delta y}\right) + A\left(h\left(T_\infty - T_{m,n}\right)\right) = 0$$

Since each node is equally spaced, Δx =Δy and multiplying the equation by 2,

$$-\frac{Ak}{\Delta x}\left(\left(T_{m,n} - T_{m-1,n}\right) + \left(T_{m,n} - T_{m+1,n}\right) + 2\left(T_{m,n} - T_{m,n-1}\right) + 2\frac{h\Delta x}{k}\left(T_{m,n} - T_\infty\right)\right) = 0$$

Rearranging,

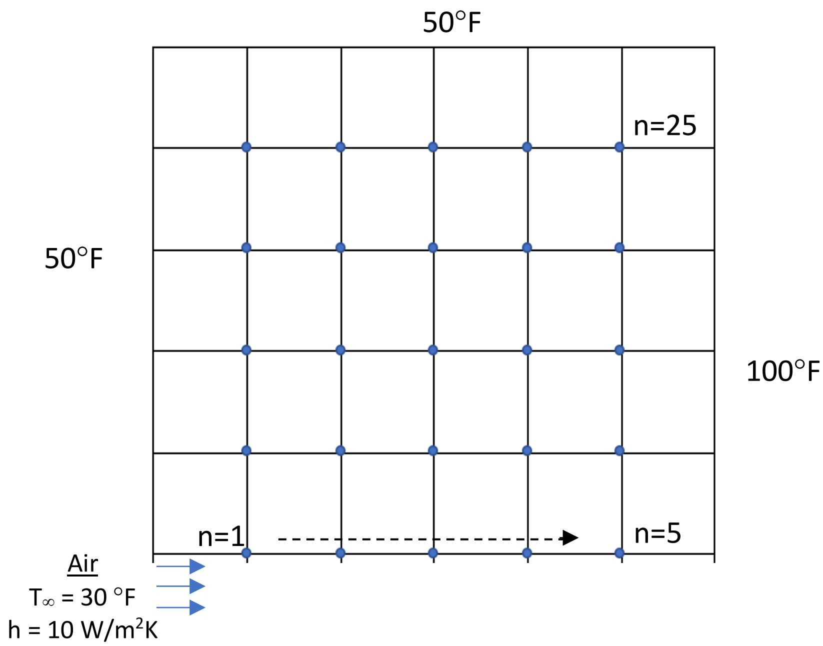
$$\left(4 + \frac{2h\Delta x}{k}\right)T_{m,n} - 2T_{m,n-1} - T_{m+1,n} - T_{m-1,n} = \frac{2h\Delta x}{k}T_\infty$$

Another useful application is when the boundary is insulated, so there is no heat transfer from the plate to the surroundings. This occurs when the heat transfer coefficient, h, is equal to 0. The equation for an insulated boundary condition would then be expressed as

$$4T_{m,n} - 2T_{m,n-1} - T_{m+1,n} - T_{m-1,n} = 0$$

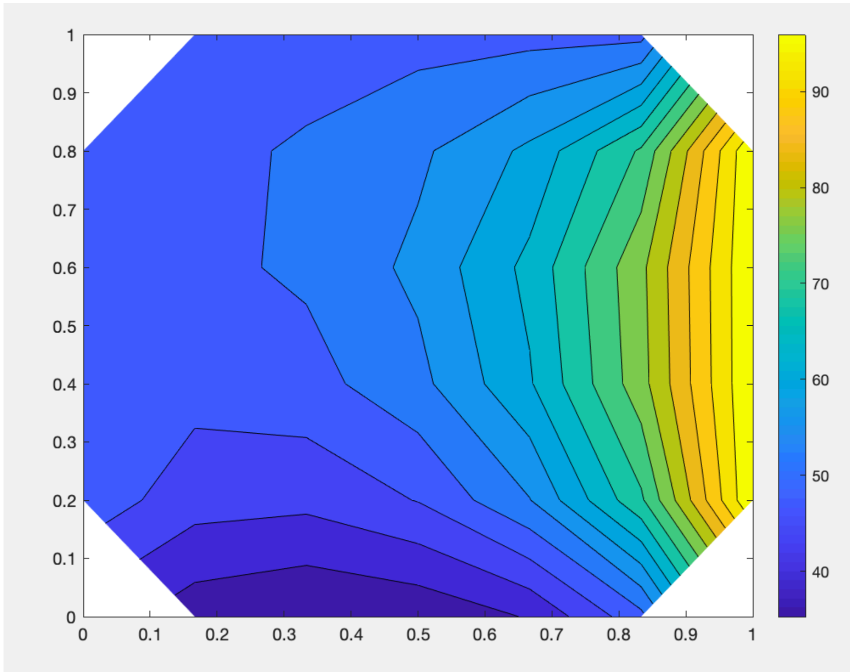
Convection Across One Surface, Prescribed Temperatures on Other Surfaces

Let k = 1 W/mK for a heated plate with n = 5, Δx = 0.2 m:



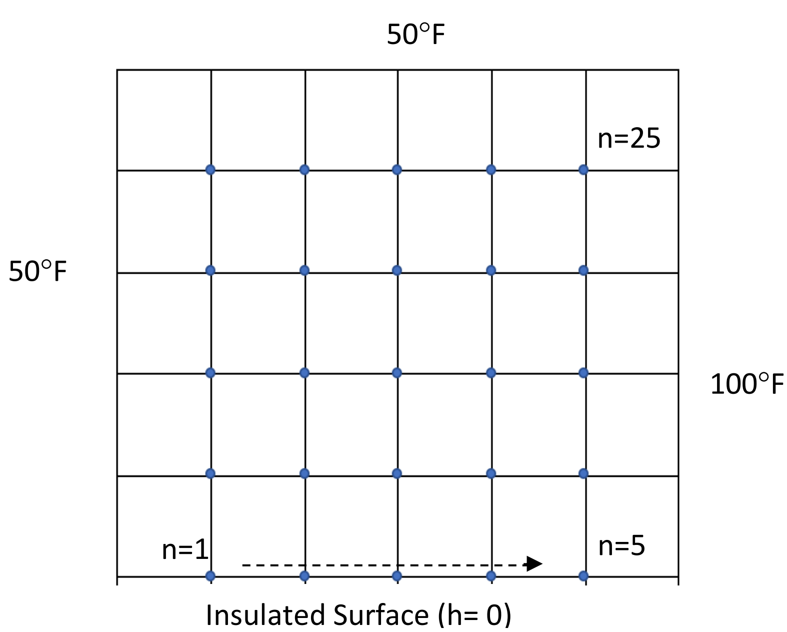
The augmented matrix [A|b] for the set of nodes, with one side having convection is expressed below.

8	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	170
-1	8	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	120
0	-1	8	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	120
0	0	-1	8	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	120
0	0	0	-1	8	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	220
-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	100
0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	50
0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	100
0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	50
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	150



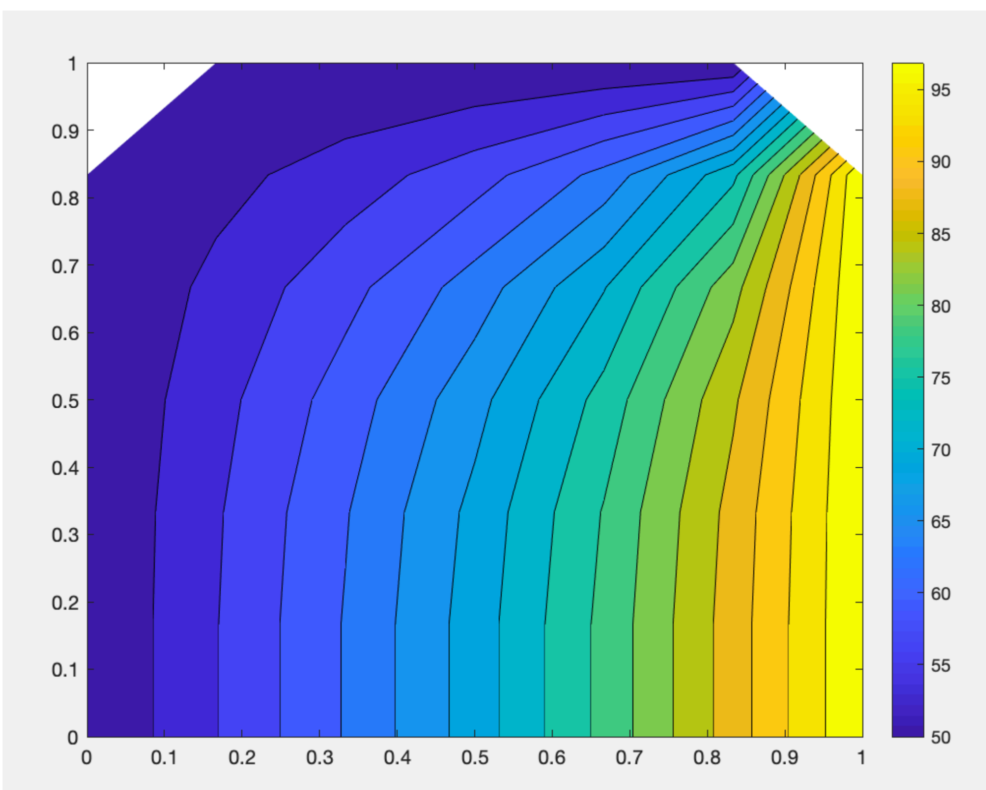
From the new contour plot, it can be observed that the side with convection has an effect on more nodes than the prescribed temperatures case of a heated plate. More heat is transferred on this plate with convection to the surroundings, rather than the prescribed temperatures. However, the general shape of the contours is similar to the plate with prescribed temperatures. The biggest difference is that as n increases along the bottom of the plate, the temperature increases, whereas it was a constant temperature in the prescribed temperature case.

<u>Insulation Across One Surface, Prescribed Temperatures on Other Surfaces</u>



The augmented matrix [A|b] for the set of nodes, with one side being insulated is expressed below.

4	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
-1	4	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	4	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	4	-1	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	4	0	0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100
-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	50
0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	100
0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	50
0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	0	0	0	0	0	100
0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	-1	0	0	0	0	50
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	0	0	0	0	-1	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	4	-1	0	0	0	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	-1	50
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	-1	4	150



Since there is insulation, there is no heat transfer along the bottom surface, so the temperature increases significantly along the bottom of the plate. In the case of constant temperature, the temperature had a less significant gradient along the bottom of the plate.

As seen from these different cases, linear algebra techniques can be used for a variety of boundary conditions, and result in different temperature gradients throughout the plate.