

Fractals: Their Composition and Beauty

Joshua Jonas

University of Utah

Abstract

In this paper I will be discussing fractals. Fractals are patterns that repeat themselves endlessly at smaller and smaller scales, and they can be incredibly beautiful and even useful in some cases. I will give a brief overview of what fractals are and how they are usually formed. I will then discuss the Koch Snowflake and Sierpinski's Triangle. Following that, I will be looking further in depth at escape time fractals, a type of fractal that is incredibly easy to compute in Maple. Specifically, I will be discussing the Mandelbrot Set and the Julia Set. I will then finish my discussion of fractals by looking at how fractals appear in some familiar places in nature.

Introduction:

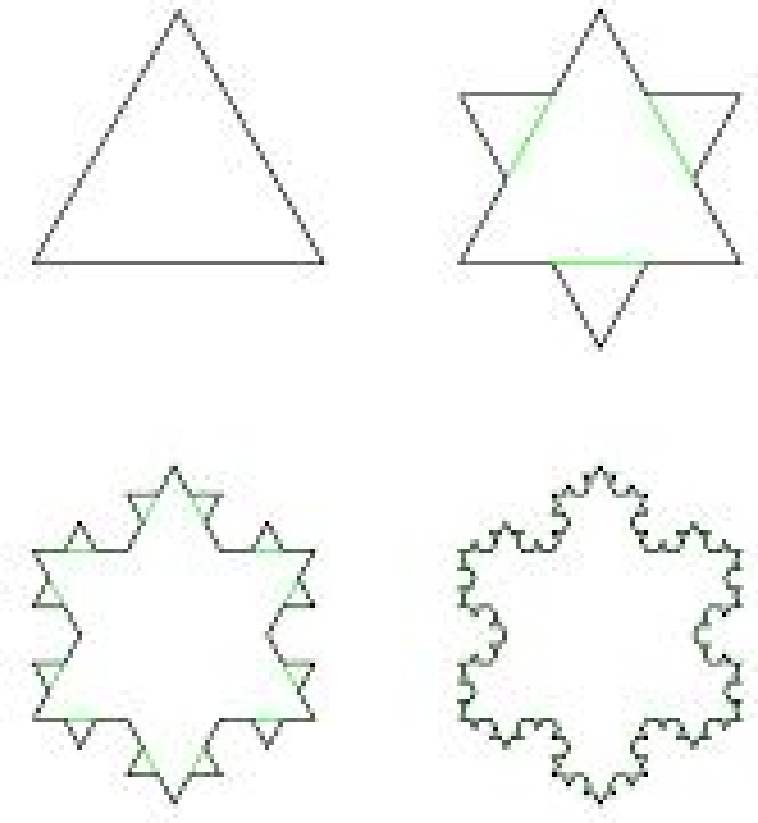
Fractals are patterns that exhibit self similarity at any scale. In other words, if you zoom in on a specific part of a fractal you will be able to see that it repeats the pattern you saw earlier. Fractals are extremely complex, and their self similarity is theoretically infinite. However, despite their complexity fractals are relatively easy to create and manipulate thanks to many modern programs we have. Fractals can be computed in a variety of ways, but almost all algebraic computations stem from some form of an iterated function system.

Iterated Function Systems:

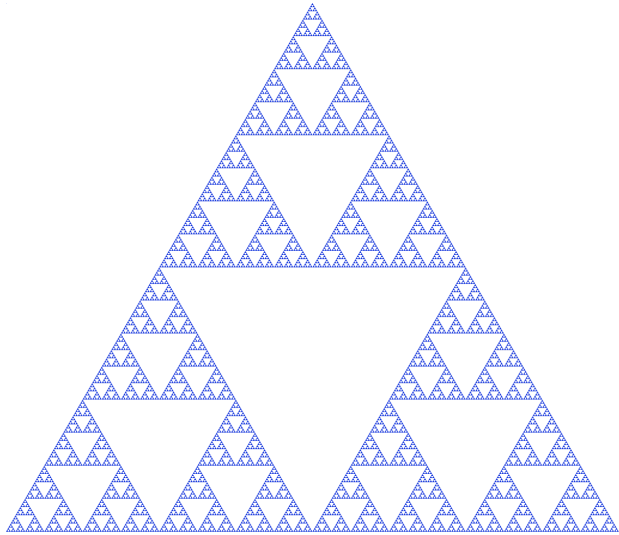
All fractals can be made through some form of iterated function system. Iterated function systems are simply a function that is applied repeatedly to a certain x in its domain. A transformation or equation is performed on a certain x , then the same transformation or equation is performed on the output. In the case of fractals, this system repeats infinitely and the x in the domain is a picture.

Koch Snowflake and Sierpinski Triangle:

Some of the most well known fractals, such as the Koch Snowflake and Sierpinski's Triangle are often made geometrically. The Koch Snowflake starts as a simple, equilateral triangle. Smaller equilateral triangles are then added to every side while the line separating the structures is erased. The process is repeated infinitely to produce the snowflake-like structure we are familiar with.



Sierpinski's Triangle is created geometrically in a very similar way. To create Sierpinski's Triangle, one starts with a solid equilateral triangle facing point up. Then, an upside down, equilateral triangle is taken from the center so that the points of the upside down triangle are at the midpoints of the sides of the initial equilateral triangle. The process is repeated with the solid, point up triangles that remain infinitely. (An interesting note: an approximation to the Sierpinski Triangle can be made by taking Pascal's triangle and coloring all of the even numbers white and the odd numbers black.)



We can also understand the composition of these two fractals in terms of linear algebra. As we are familiar with, affine transformations can be used to rotate, translate, and scale different objects or pictures. These two fractals can be understood in terms of translation, rotation, and scaling as well. In both instances, affine transformations can be used to scale down and/or rotate the initial picture. Then that scaled down or rotated version is placed inside of, added on to, or replaces the additional shape. In the case of the Koch Snowflake, the original triangle is scaled down, rotated, and placed on one of the edges of the original triangle. The process is repeated for the other two sides. The triangle is then scaled down more and placed on the edges of the new triangles. This process is repeated infinitely. In the case of Sierpinski's triangle, the original triangle is scaled down, and stacked in a pyramidal form with three components. This process is repeated with the remaining point up facing triangles infinitely.

Affine Transformations:

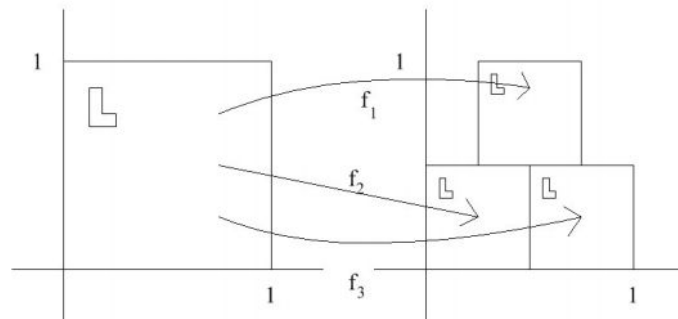
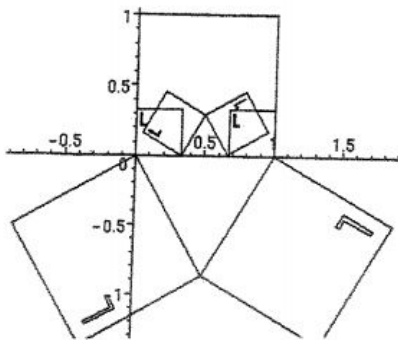
To better understand what we have discussed above, let's examine affine transformations in more detail. Generally speaking, affine transformations have the following form.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

In this case, a, b, c, d, e, and f are scalars. The 2x2 matrix can represent a variety of transformation matrices that can rotate, scale, or perform some other operation. We can manipulate the variables in this equation to make changes to the original picture. Then we can join the original shape with the new shape we have transformed using an affine transformation to create fractals such as the ones above.

Fractal Templates:

The concept of using affine transformations to create fractals can sometimes be better understood with a picture of the template with which the fractal is made. Below are the templates used to create the Koch Snowflake and Sierpinski's Triangle, respectively.



Escape Time Fractals:

Another fractal that can be created using iterated function systems is an escape time fractal. In an escape time fractal, a formula is iterated multiple times, and the points that stay bounded remain in the frame while the points that diverge leave. Often times how quickly each point leaves is illustrated by a certain shading of the picture. There are multiple different escape

time fractals that Maple can be used to create, but for the sake of brevity I will only go over the Mandelbrot Set and the Julia Set.

Mandelbrot Set:

The Mandelbrot set uses complex values and is iterated using the following formula on each point c :

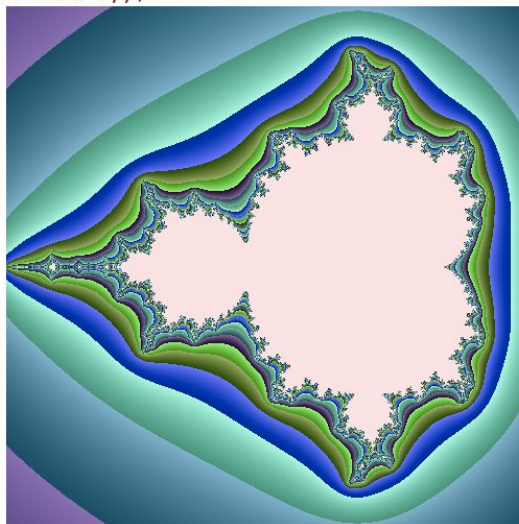
$$z \rightarrow z^2 + c, z_0 = 0$$

We start by plugging some complex variable c into the equation above. We then run the formula and come out with a new z (call it z_{new}). We then run the formula with z_{new} and come out with a new z . This process is repeated infinitely. In most cases, z tends to infinity and escapes.

However, for some values of c the value of z remains bounded. These values c are what make up the Mandelbrot Set.

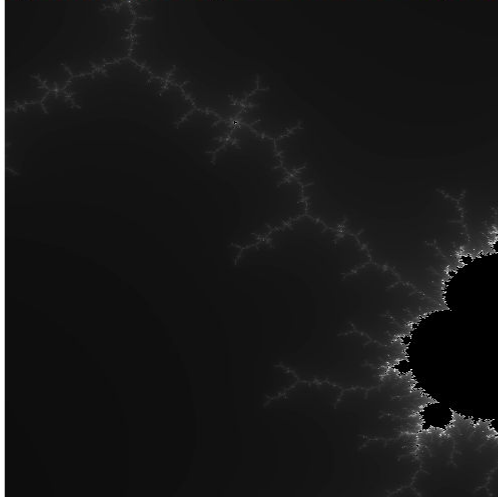
The Mandelbrot set is incredibly easy to create in Maple 2019. The following code creates a beautiful picture that reminds me of some prehistoric life form.

```
> with(Fractals:-EscapeTime):  
> with(ImageTools):  
>  
> Embed(Mandelbrot(500,-2.0-1.35*I,0.7+1.35*I));
```



It is also easy to play around with the values and options to create other interesting fractals. I created the fractal below in no time by modifying the values in the equation used to create the first picture and changing the output value to black and white.

```
|> Embed(Mandelbrot(500,-0.1313+1.011*I,-0.1575+1.043*I,iterationlimit=200,cutoff=50.0,output=layer1));
```



Julia Set:

The Julia Set is created in a very similar way to the Mandelbrot Set. The Julia Set uses the same formula $z \rightarrow z^2 + c$, $z_0 = 0$, but the iteration is different. For a fixed parameter c , the set is found by iterating the formula on each point z .

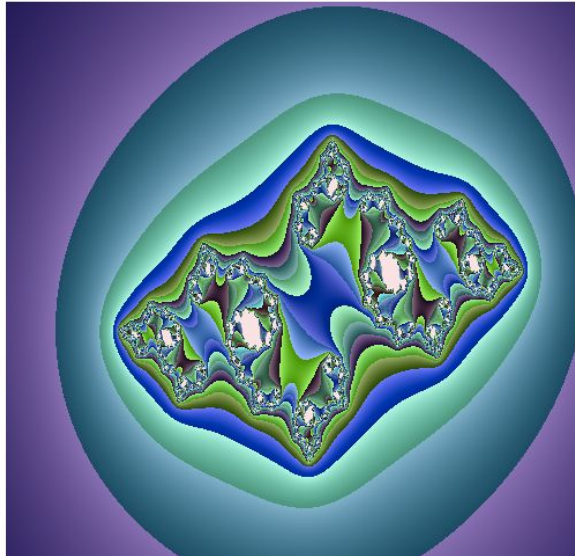
The Julia Set is also very easy to create using maple. The following code can be used to create a Julia Set that looks to me like an image from a kaleidoscope.


```
> with(Fractals:-EscapeTime):  
> with(ImageTools):  
> bl:=-2.4-1.6*I  
  
> ur:=2.0+1.5*I  
  
> c:=-0.8+0.314*I  
  
> Embed(Julia(500,bl,ur,c));
```

$bl := -2.4 - 1.6I$

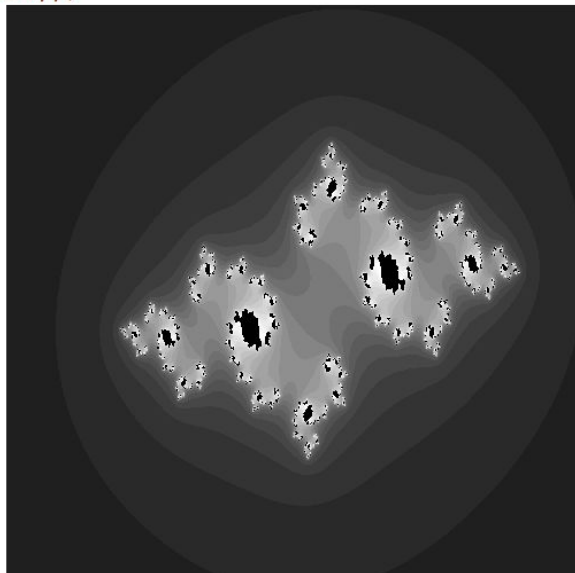
$ur := 2.0 + 1.5I$

$c := -0.8 + 0.314I$



This set, too, is extremely easy to manipulate. I created the following image by simply changing the output to black and white. The image looks to me like two galaxies about to collide.

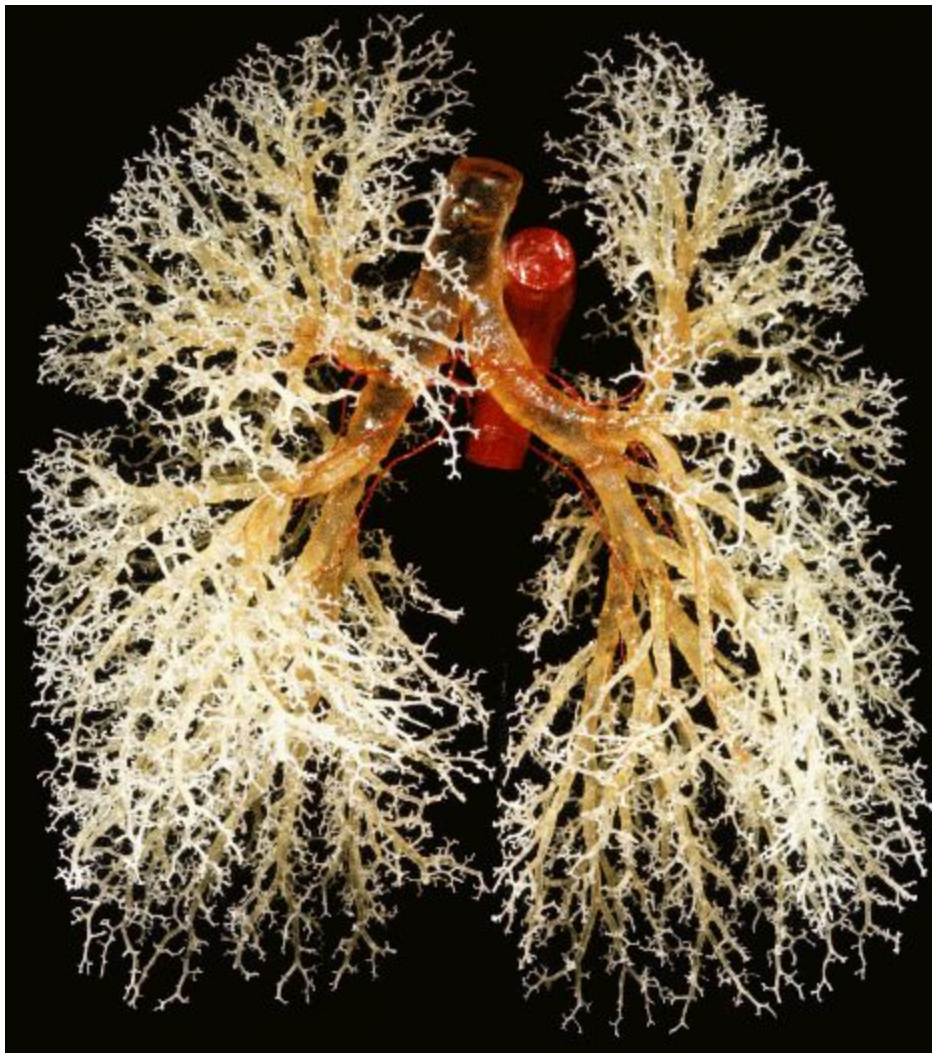
```
[> Embed(Julia(500,bl,ur,c,output=layer1));
```



Fractals in Nature:

One of the most interesting things about fractals is how often they seem to appear in nature. River deltas, galaxies, and even parts of the human body often show the same sort of self-similarity that mathematically created fractals do. By looking at pictures of these instances we can see just how well mathematics is able to imitate the world.

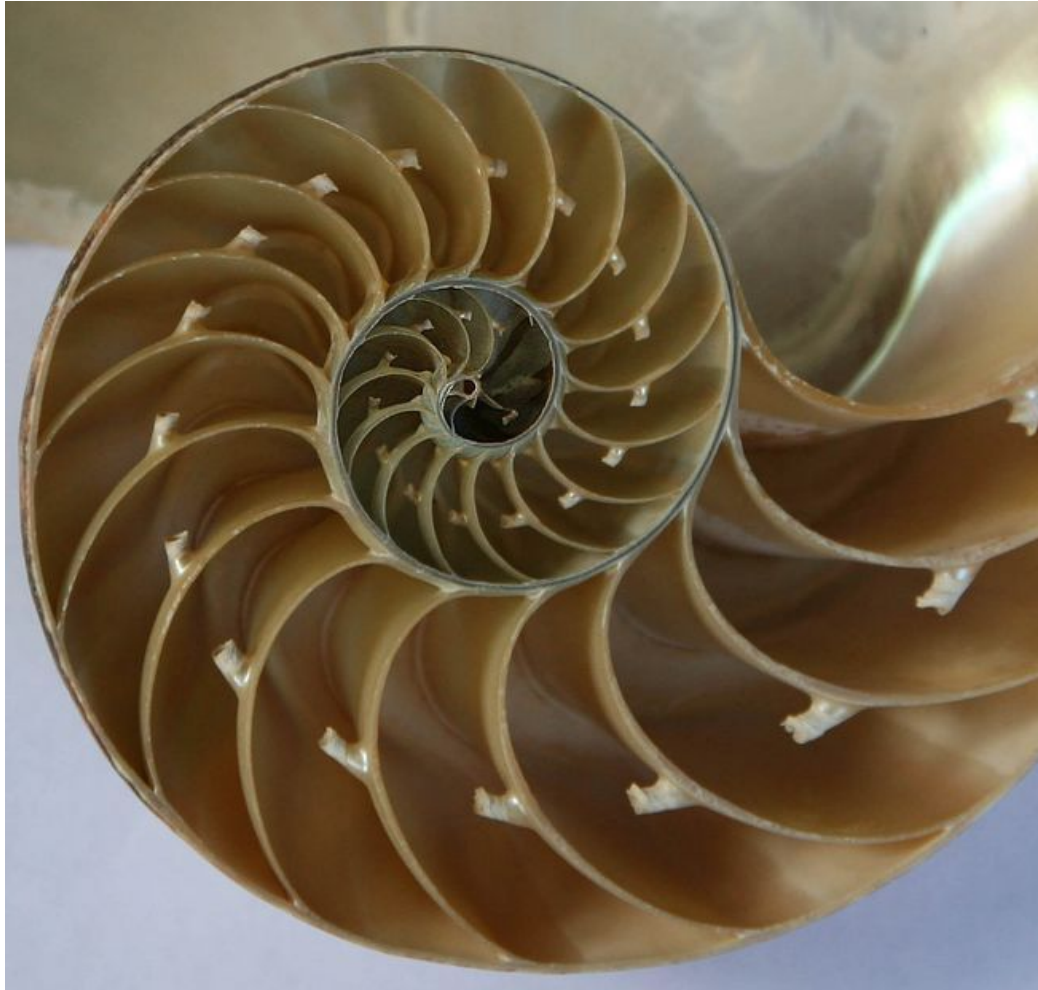
Below we can see an image of a cast of the human lungs. As is apparent, they exhibit remarkable self-similarity. Each smaller branch seems to resemble the larger branch it is a part of.



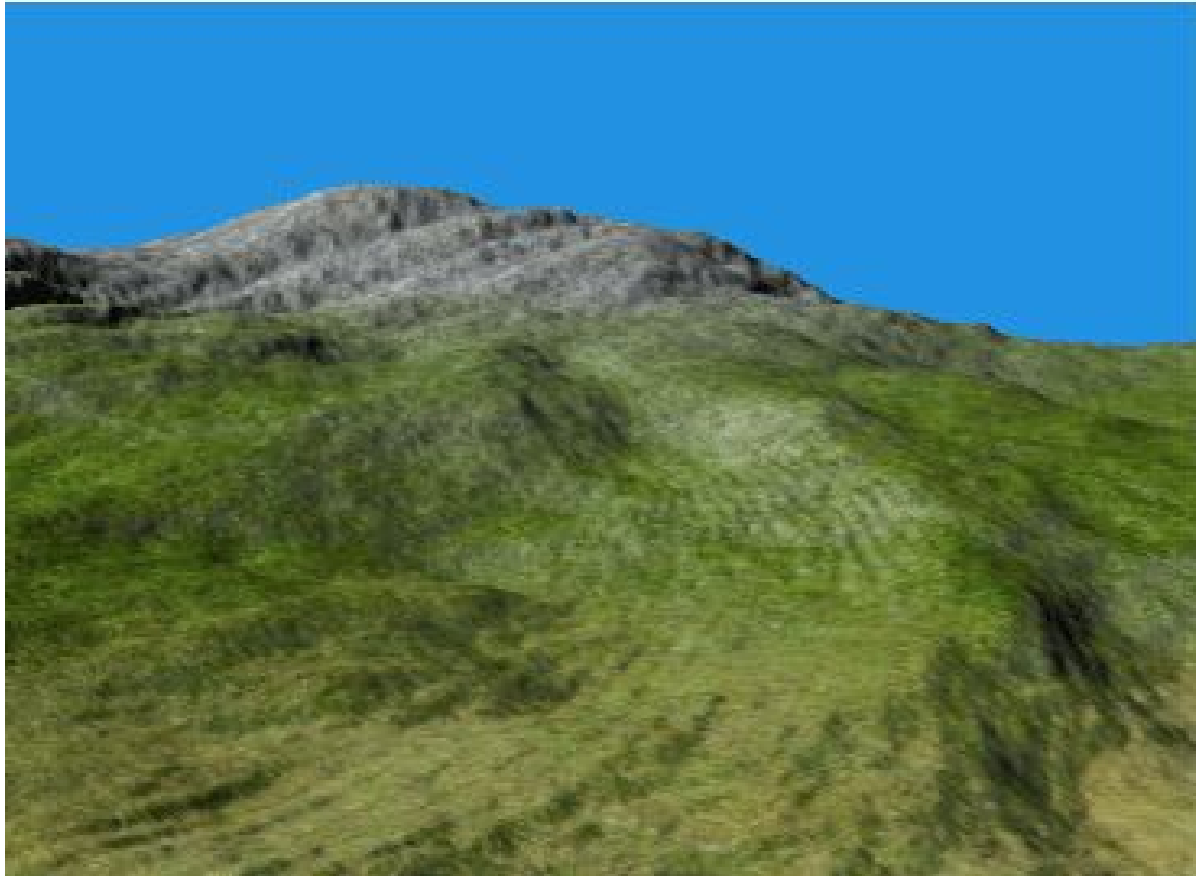
Another interesting example of fractals in nature is this fern plant. As you can see, each sprout off of the initial stalk looks like a smaller version of the whole.



This chambered nautilus shell is another example of a fractal in nature. If you zoom directly into the center of the spiral the pattern will be the same.



As is expected, mathematics' ability to imitate nature in this field has been utilized by many. Many CGI landscapes are generated using fractals, such as the one below.

**Conclusion:**

As we can see, fractals are incredibly beautiful and complex, and they often times seem to imitate certain parts of nature. This begs the question, are there parts of nature we have yet to discover that are already imitated by a fractal we have mathematically created? Maybe somewhere in the cosmos there is a material version of the Mandelbrot Set or some other fractal. We just have to find it.

References

Escape Time Fractals. *MapleSoft*. Retrieved from

<https://www.maplesoft.com/support/help/Maple/view.aspx?path=MathApps/EscapeTimeFractals>

Fractal Pack 1 Educators' Guide. (2009). *Fractal Foundation*. Retrieved from

<https://fractalfoundation.org/fractivities/FractalPacks-EducatorsGuide.pdf>

Gunther, Shea. 14 Amazing Fractals Found in Nature. (2013). *Mother Nature Network*.

Retrieved from

<https://www.mnn.com/earth-matters/wilderness-resources/blogs/14-amazing-fractals-found-in-nature>

Klang, Jesse. (2004). An Introduction to Fractals. Retrieved from

<http://home2.fvcc.edu/~dhicketh/LinearAlgebra/studentprojects/fall2004/jesseklang/Fractalproject.htm>

Korevaar, N. FractalsMaple13 [PDF document]. Retrieved from Lecture Notes Online Web site:

<http://www.math.utah.edu/~korevaar/fractals/>

Korevaar, N. IFSFractals [PDF document]. Retrieved from

<http://www.math.utah.edu/~korevaar/fractals/IFSfractals.pdf>

Schneider, Jens. Real-Time Editing, Synthesis, and Rendering of Infinite Landscapes on GPUs.

(2012). *ResearchGate*. Retrieved from

https://www.researchgate.net/figure/A-fractal-landscape-generated-by-our-method-Note-the-striking-geological-features-and_fig1_228909493

Taylor, Richard. Fractals in Nature are Aesthetically Pleasing and Stress Reducing. (2017).

Smithsonian Magazine. Retrieved from

<https://www.smithsonianmag.com/innovation/fractal-patterns-nature-and-art-are-aesthetically-pleasing-and-stress-reducing-180962738/>

Toxicology of the Lung. University of Washington. Retrieved from

<http://depts.washington.edu/envh/lung.html>