Discrete Dynamic Systems That Approximate Motion Through Velocity and Acceleration Vector Fields: Using Jacobian Matrices to Improve the Accuracy of the Approximation.

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5/2/2017

Some multivariable calculus required

Contents:

Introduction:

Sometimes discrete dynamic systems are used to approximate continuous problems like solutions to some differential equations and kinematics especially if it cannot be solved exactly. This project will focus on motion through vector fields that represent either velocity or acceleration. These can be approximated by the discrete dynamic systems:

For velocity fields:

$$\boldsymbol{v}_t = \boldsymbol{f}(\boldsymbol{r}_t)$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t$$

For acceleration fields:

$$\boldsymbol{a_t} = \boldsymbol{f}(\boldsymbol{r_t})$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t$$

$$\boldsymbol{v}_{t+\Delta t} = \boldsymbol{v}_t + \Delta t \boldsymbol{a}_t$$

Where the continuous system is the limit as Δt approaches zero. Making Δt smaller will make the model more accurate but require more computation to simulate the system for a certain time. There are other ways to approximate these more accurately (other than exact methods). This project will try to find some that involve using matrices to perform operations on these fields.

How the results are generated:

All graphs in this project are the result of numerical calculations done in excel. This is done by columns of cells that reference members of the other columns of a group representing parameters of the system in the row above or on the same level.

Velocity vector fields:

$$\boldsymbol{v}_t = \boldsymbol{f}(\boldsymbol{r}_t)$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t$$

One way to make any approximation of next position more accurate is similar to a second order Taylor approximation:

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t + \frac{1}{2}\Delta t^2 \boldsymbol{a}_t$$

The only difficulty is finding the acceleration since no value is given. This is how linear algebra is used. Matrices multiplication allow for new ways to manipulate vectors and vector fields. For a vector field, the Jacobian matrix can be used to determine the vector differential of the field (Notated by D, referred to in some of the graphs as a d-matrix):

$$D_v = \begin{bmatrix} (\nabla v_{x1})^T \\ (\nabla v_{x2})^T \\ \vdots \end{bmatrix}$$

(From calculus: $\nabla v_{xn} = \begin{matrix} \frac{\partial v_{xn}}{\partial x_1} \\ \frac{\partial v_{xn}}{\partial x_2} \\ \cdots \end{matrix}$)

This can be used to find the differential of **dx** with respect to any infinitesimal displacement vector by multiplication of D by **dx**. Why this works is shown below:

$$d\mathrm{F} = \nabla F \cdot d\boldsymbol{r}$$

So: $$d\mathrm{v}_{x1} = \nabla v_{x1} \cdot d\boldsymbol{r} = (\nabla v_{x1})^T d\boldsymbol{r}$$

And $$d\mathrm{v}_n = (\nabla v_{xn})^T d\boldsymbol{r}$$

When combined: $$d\boldsymbol{v} = \begin{bmatrix} (\nabla v_{x1})^T d\boldsymbol{r} \\ (\nabla v_{x1})^T d\boldsymbol{r} \\ \vdots \end{bmatrix} = \begin{bmatrix} (\nabla v_{x1})^T \\ (\nabla v_{x2})^T \\ \vdots \end{bmatrix} d\boldsymbol{r} = D_v d\boldsymbol{r}$$

For determining the acceleration, $d\boldsymbol{r} = \boldsymbol{v}dt$

So: $$d\boldsymbol{v} = D_v \boldsymbol{v} dt$$

Thus the acceleration of the field at any position is given by:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = D_v \boldsymbol{v}$$

Which gives:

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t + \frac{1}{2}\Delta t^2 D_v \boldsymbol{v}_t$$

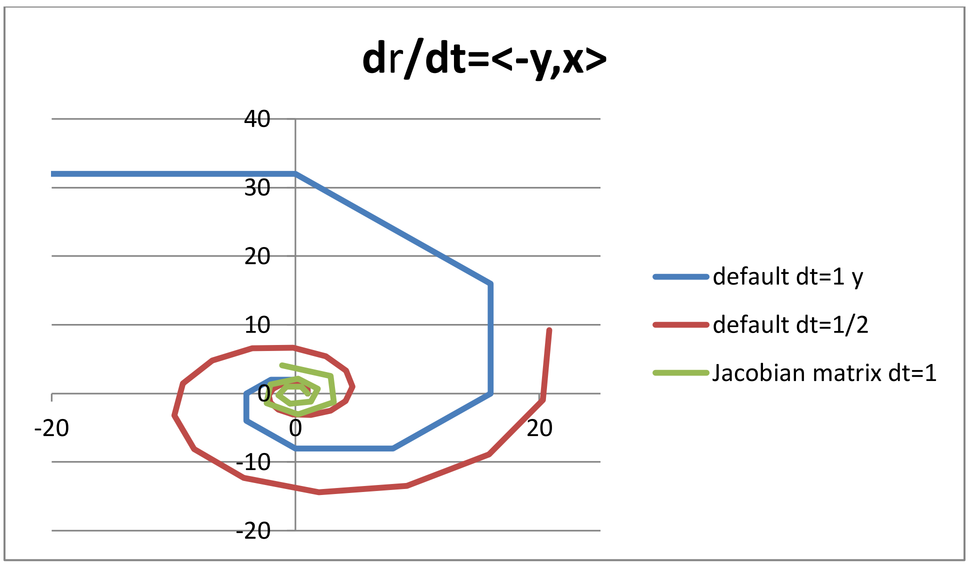
Example field in two dimensions: $\boldsymbol{v} = \begin{pmatrix} -y \\ x \end{pmatrix}$

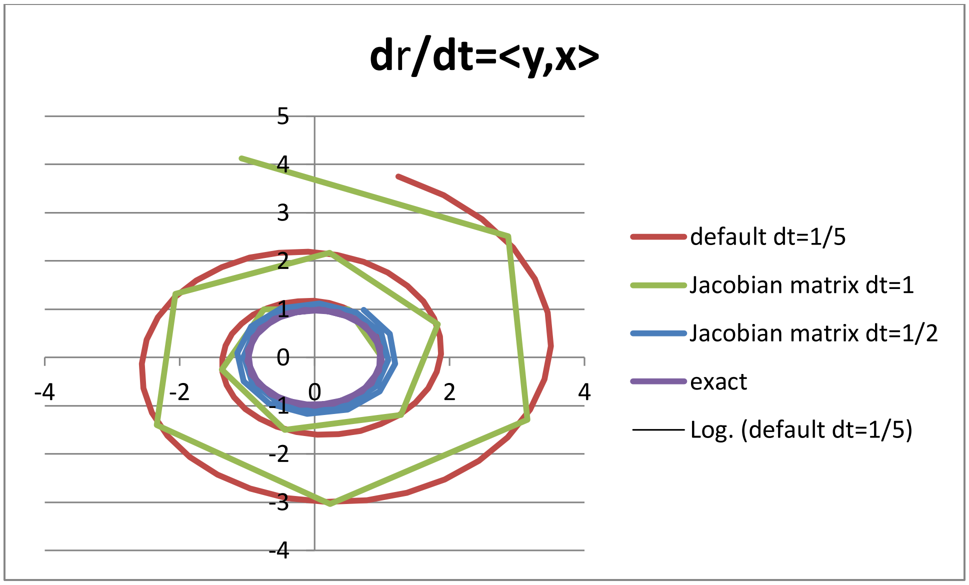
$$D_v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$D_v \boldsymbol{v} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t - \frac{1}{2}\Delta t^2 \boldsymbol{r}_t$$

(Note: all the example fields happen to be linear dynamic systems because they are easier to enter into excel. This is not intended to be the case for all uses.)





Higher orders:

The Jacobian matrix can be used to determine higher position based derivatives as well:

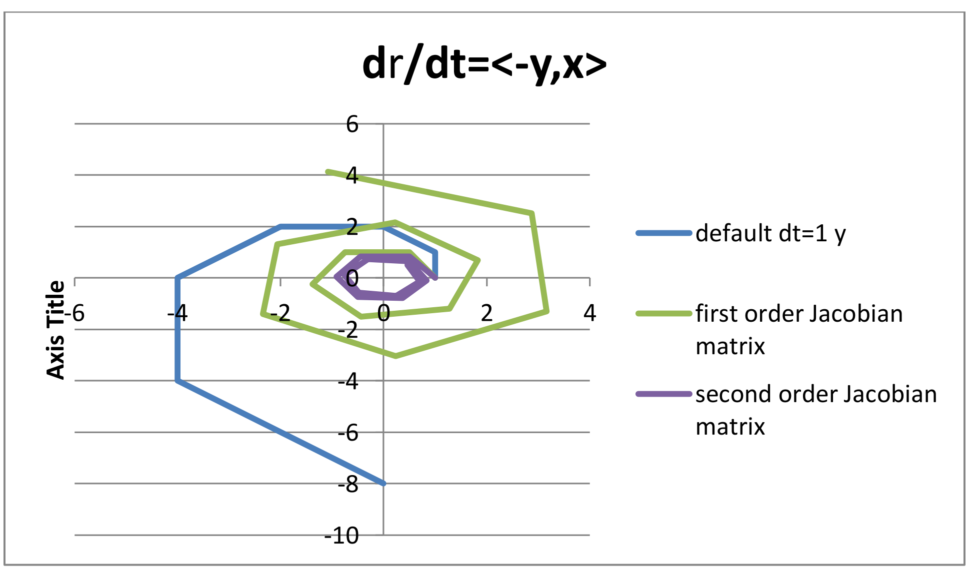
$$d\boldsymbol{a} = D_a d\boldsymbol{r} = D_a \boldsymbol{v} dt$$

$$\frac{d\boldsymbol{a}}{dt} = D_a \boldsymbol{v}$$

The higher orders of the approximation can then be applied:

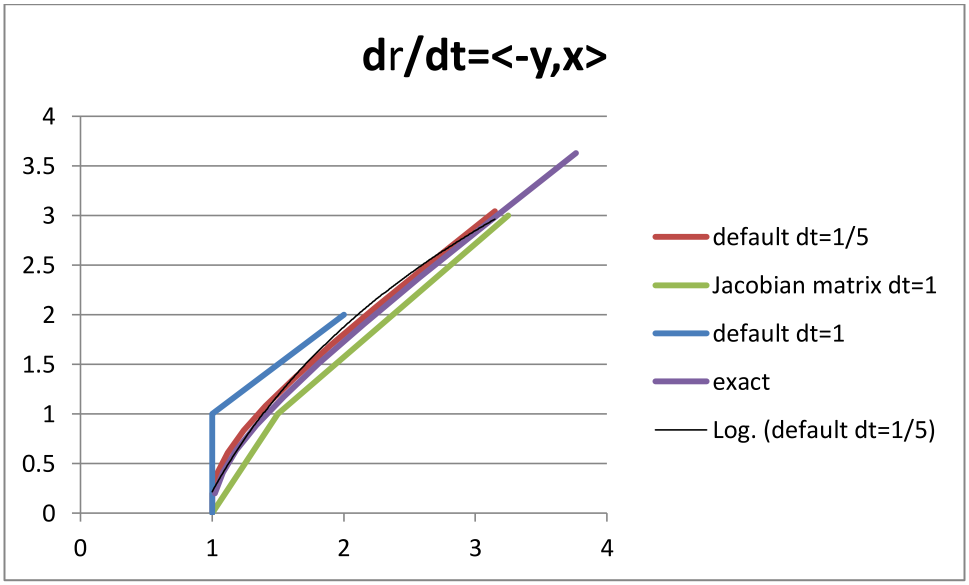
$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t + \frac{1}{2}\Delta t^2 D_v \boldsymbol{v}_t + \frac{1}{6}\Delta t^3 D_a \boldsymbol{v}_t + \frac{1}{24}\Delta t^4 D_{da/dt} \boldsymbol{v}_t + \cdots$$

The second order of this has been applied to the previous vector field:



The second order appears to be even more accurate.

Another example field: $\boldsymbol{v} = \binom{y}{x}$



Results:

This Jacobian matrix method appears to work well for velocity vector fields.

Acceleration vector fields:

$$\boldsymbol{a_t} = \boldsymbol{f}(\boldsymbol{r_t})$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t$$

$$\boldsymbol{v}_{t+\Delta t} = \boldsymbol{v}_t + \Delta t \boldsymbol{a}_t$$

The Jacobian matrix method from velocity vectors can be used here to adjust the velocity term:

$$d\boldsymbol{a} = D_a d\boldsymbol{r} = D_a \boldsymbol{v} dt$$

$$\frac{d\boldsymbol{a}}{dt} = D_a \boldsymbol{v}$$

$$\boldsymbol{v}_{t+\Delta t} = \boldsymbol{v}_t + \Delta t \boldsymbol{a}_t + \frac{1}{2}\Delta t^2 D_a \boldsymbol{v}_t$$

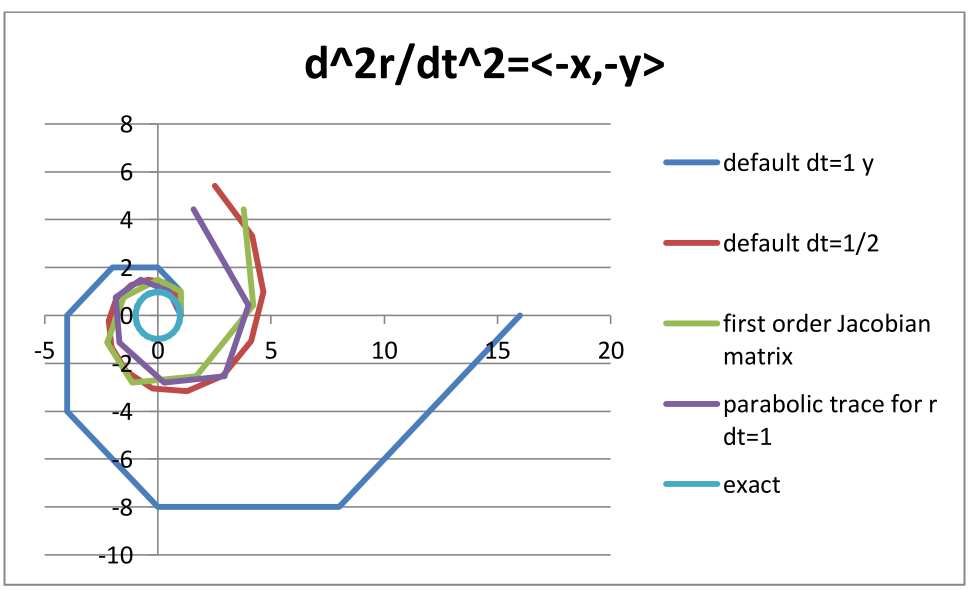
The position can also be better approximated by taking into account the effect of acceleration:

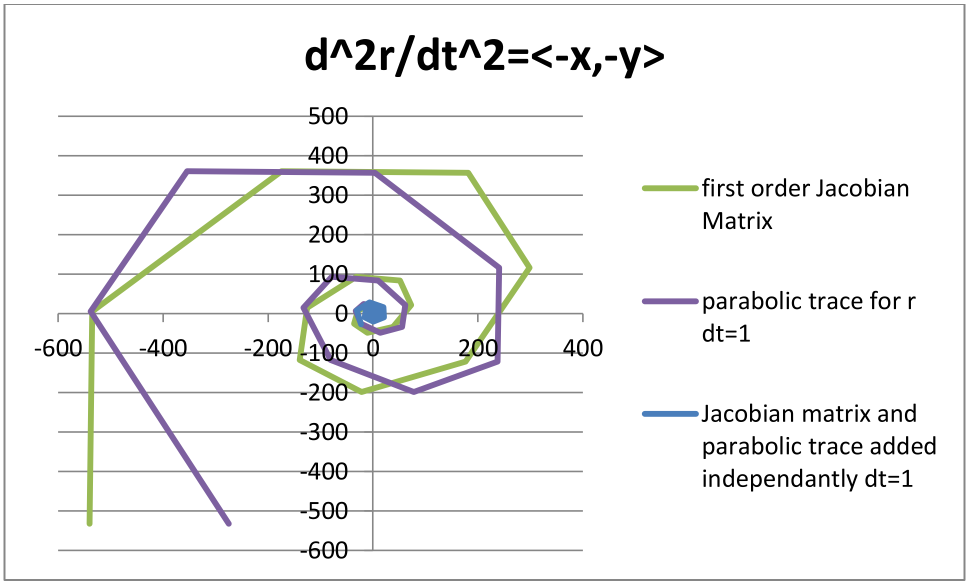
$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t + \frac{1}{2}\Delta t^2 \boldsymbol{a}_t$$

This time, the acceleration is given. One or both of the methods above can be used:

Example field: $\boldsymbol{a} = \binom{-x}{-y}$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$





Results: both methods make the model more accurate but using both methods is more effective.

$$\boldsymbol{v}_{t+\Delta t} = \boldsymbol{v}_t + \Delta t \boldsymbol{a}_t + \frac{1}{2}\Delta t^2 D_a \boldsymbol{v}_t$$

$$\boldsymbol{r}_{t+\Delta t} = \boldsymbol{r}_t + \Delta t \boldsymbol{v}_t + \frac{1}{2}\Delta t^2 \boldsymbol{a}_t$$

Conclusion:

The Jacobian matrix method appears to work well for approximating paths through velocity vector fields.

Both the Jacobian matrix and parabolic trace increase the accuracy for acceleration vector fields by about the same amount and this effect is compounded when both are used.