

**> # Maple Lab 6: Fibonacci and Lucas Sequences**  
**> with(LinearAlgebra):**

**▼ Problem 1.**

Define  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ . Using Maple, compute  $A^5$  and use it to find vector  $u_5$  and then find Fibonacci number  $y_5$ .

**> y0:=0:y1:=1:A:=<0,1|1,1>;u0:=<y0,y1>;<y5,y6> = (A^5).u0;**

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$u0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y5 \\ y6 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

(1)

**▼ Problem 2.**

Show that the eigenvalues of matrix A are  $\lambda_1 = \frac{1 + \sqrt{5}}{2}$  and  $\lambda_2 = \frac{1 - \sqrt{5}}{2}$  by solving the characteristic equation of A.

**> chareq:=CharacteristicPolynomial(A,lambda);  
lambda1,lambda2:=solve(chareq,lambda);**

$$chareq := \lambda^2 - \lambda - 1$$

$$\lambda_1, \lambda_2 := \frac{1}{2} \sqrt{5} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2} \sqrt{5}$$

(2)

**▼ Problem 3.**

Show that  $\begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$  are eigenvectors of A corresponding to  $\lambda_1$  and  $\lambda_2$  respectively.

Computations will use  $\lambda_1 + \lambda_2 = 1$  and  $\lambda_1 \lambda_2 = -1$ .

**> # AP=PD verifies the eigenpairs. Check with Eigenvectors(A);**

**> P:=<-lambda2,1|-lambda1,1>;DD:=<lambda1,0|0,lambda2>;**

$$P := \begin{bmatrix} -\frac{1}{2} + \frac{1}{2} \sqrt{5} & -\frac{1}{2} \sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

(3)

$$DD := \begin{bmatrix} \frac{1}{2}\sqrt{5} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix} \quad (3)$$

> **simplify(A.P-P.DD); # AP=PD verifies the eigenpairs**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4)$$

#### Problem 4.

A 2x2 matrix A is defined to be diagonalizable provided it has 2 eigenpairs. Explain why A is diagonalizable, by exhibiting the 2 eigenpairs.

> **# TO DO: Compute eigenpairs. See problem 3 code.**

> **# TO DO: Explain why A is diagonalizable.**

#### Problem 5.

A diagonalizable matrix A satisfies the equation  $AP=PD$  where D is a diagonal matrix of eigenvalues and P is the augmented matrix of corresponding eigenvectors (D and P are not unique). Use the results of Problem 4 to find an invertible matrix P and a diagonal matrix D for which  $A = PDP^{-1}$ .

> **# Use AP=PD diagonalization theory from the textbook.**

#### Problem 6.

Use Maple to calculate  $D^{10}$ , then use it to find  $A^{10} = PD^{10}P^{-1}$ , vector  $\mathbf{u}_{10}$ , and Fibonacci number  $y_{10}$ . Confirm your result for  $y_{10}$  by writing out the Fibonacci sequence by hand and then consulting Wikipedia. In the solution, explain in detail why  $A^{10} = PD^{10}P^{-1}$ .

> **P:=<-lambda2,1|-lambda1,1>;DD:=<lambda1,0|0,lambda2>;**

> **DD^10; <y10,y11> = simplify(P . DD^10 . (1/P) . u0);**

$$\begin{bmatrix} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^{10} & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^{10} \end{bmatrix}$$

$$\begin{bmatrix} y10 \\ y11 \end{bmatrix} = \begin{bmatrix} 55 \\ 89 \end{bmatrix} \quad (5)$$

> **# AP=PD explains why A^10 = P D^10 (1/P)**

#### Problem 7.

A formula for  $y_k$  may be derived using the preceding two problems. Use already derived expressions

for  $P$ ,  $D$ , and  $A^k$  to obtain the formula for  $k=1,2,3, \dots$

```
> Ak := 1/sqrt(5) * Matrix( 2,2, [ [ lambda[2]^k*lambda[1], lambda
[2]^(k+1)*lambda[1]-lambda[1]^(k+1)*lambda[2] ], [lambda[1]^k-
lambda[2]^k, lambda[1]^(k+1)-lambda[2]^(k+1) ] ] ):
print( Ak );
```

$$\begin{bmatrix} \frac{1}{5} \sqrt{5} \lambda_2^k \lambda_1 & \frac{1}{5} \sqrt{5} (\lambda_2^{(k+1)} \lambda_1 - \lambda_1^{(k+1)} \lambda_2) \\ \frac{1}{5} \sqrt{5} (\lambda_1^k - \lambda_2^k) & \frac{1}{5} \sqrt{5} (\lambda_1^{(k+1)} - \lambda_2^{(k+1)}) \end{bmatrix}$$

(6)

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> # TO DO: Reformulate P, DD with symbols c,d. Then compute
# power k of A from the formula AP=PD.
```

### Problem 8.

Use the result of Problem 7 to find vector  $u_k$  and Fibonacci number  $y_k$ . Simplifications use  $cd = \lambda_1 \lambda_2 = -1$ . The answer should be

```
> u0 := Vector( 2, [1,1] ):
uk := Ak . u0:
yk := uk[1]:
yk;
```

$$y_k = \frac{1}{\sqrt{5}} (\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right)$$

Calculate Fibonacci number  $y_{10}$  using this formula, and compare your result to that of Problem 6.

### Problem 9.

The second eigenvalue  $\lambda_2$  is less than 1 in absolute value, therefore term  $\lambda_2^k$  will approach zero as  $k$  approaches infinity. The following approximate equation results for the  $k^{\text{th}}$  Fibonacci number:

$$y_k \sim \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k$$

Use this approximation to approximate  $\frac{y_{k+1}}{y_k}$ .

### Problem 10.

Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 3y_{k+1} - 2y_k$$

with  $y_0 = 0$  and  $y_1 = 1$ . Write out by hand the first seven terms of this sequence and see if you can find the pattern. Then apply eigenanalysis, as was done for the Fibonacci sequence, to find a formula for  $y_k$

### Problem 11.

Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 2y_{k+1} - y_k$$

with  $y_0 = 0$  and  $y_1 = 1$ . Find the pattern by writing out as many terms in the sequence as you need.

Will the Fibonacci sequence analysis work in this case? Why or why not? Consult the Wikipedia page on the Lucas Sequence for a solution formula.