Exam 3 KEY 2270-4

Definition: An *abstract* vector space V is a data set of packages called **vectors** together with operations of addition (+) and scalar multiplication ($\cdot$) satisfying the following eight (8) rules:

Closure: If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in V, then $\vec{\mathbf{x}}+\vec{\mathbf{y}}$ is defined and in V.

(1) $\vec{\mathbf{x}}+\vec{\mathbf{y}}=\vec{\mathbf{y}}+\vec{\mathbf{x}}$

(2) $\vec{\mathbf{x}}+(\vec{\mathbf{y}}+\vec{\mathbf{z}})=(\vec{\mathbf{x}}+\vec{\mathbf{y}})+\vec{\mathbf{z}}$

(3) There is a zero vector $\vec{0}$ in V with $\vec{\mathbf{x}}+\vec{0}=\vec{\mathbf{x}}$.

(4) There is a vector $-\vec{\mathbf{x}}$ in V with $\vec{\mathbf{x}}+(-\vec{\mathbf{x}})=\vec{0}$.

Closure: If $c=$ constant and $\vec{\mathbf{x}}$ is in V, then $c\cdot\vec{\mathbf{x}}$ is defined and in V.

(5) $a(\vec{\mathbf{x}}+\vec{\mathbf{y}})=a\vec{\mathbf{x}}+a\vec{\mathbf{y}}$

(6) $(a+b)\cdot\vec{\mathbf{x}}=a\cdot\vec{\mathbf{x}}+b\cdot\vec{\mathbf{x}}$

(7) $(ab)\cdot\vec{\mathbf{x}}=a\cdot(b\cdot\vec{\mathbf{x}})$

(8) $1\cdot\vec{\mathbf{x}}=\vec{\mathbf{x}}$

Definition. If vectors $\vec{\mathbf{b}}_1,\vec{\mathbf{b}}_2,\ldots,\vec{\mathbf{b}}_n$ are a basis for subspace X of an *abstract* vector space V, and $\vec{\mathbf{x}}=c_1\vec{\mathbf{b}}_1+c_2\vec{\mathbf{b}}_2+\cdots+c_n\vec{\mathbf{b}}_n$ is a given linear combination of these vectors, then the uniquely determined constants $c_1,c_2,\ldots,c_n$ are called the *coordinates of $\vec{\mathbf{x}}$ relative to the basis $\vec{\mathbf{b}}_1,\vec{\mathbf{b}}_2,\ldots,\vec{\mathbf{b}}_n$* and the *coordinate map* is the isomorphism

$$\vec{\mathbf{x}}=c_1\vec{\mathbf{b}}_1+c_2\vec{\mathbf{b}}_2+\cdots+c_n\vec{\mathbf{b}}_n \rightarrow \begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}.$$

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S

(2) If vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}}+\vec{\mathbf{y}}$ is in S.

(3) If vector $\vec{\mathbf{x}}$ is in S and c is any scalar, then $c\vec{\mathbf{x}}$ is in S.

Definition: Vectors $\vec{\mathbf{v}}_1,\ldots,\vec{\mathbf{v}}_p$ in an abstract vector space V are said to be **independent** in V provided solving the equation $c_1\vec{\mathbf{v}}_1+\cdots+c_p\vec{\mathbf{v}}_p=\vec{0}$ for scalars $c_1,\ldots,c_p$ has only the zero solution $c_1=\cdots=c_p=0$.

Some problems have two solutions.

✓ **Problem 1.** **(100 points)** Let V be the vector space of all functions on $(-\infty, \infty)$. Define $W = \mathbf{span}\{x, e^x\}$. Assume known that x, e^x are independent functions. Define subspace $S = \mathbf{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ where

$$\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x.$$

(a) [20%] Explain why S is contained in W, that is, provide details for why linear combinations of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are in W.

A

Pf: Choose $\vec{v} \in S$ where $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for $c_1, c_2 \in \mathbb{R}$. We find

$$\begin{aligned}\vec{v}: y &= c_1(x+e^x) + c_2(x-e^x)\\ &= c_1 x + c_1 e^x + c_2 x - c_2 e^x\\ &= (c_1+c_2)x + (c_1-c_2)e^x\end{aligned}$$

Since $c_1+c_2,\ c_1-c_2 \in \mathbb{R}$, $\vec{v} \in W$. Therefore $S \subseteq W$. □

(b) [40%] Prove that $W = S$. Therefore $\dim(S) = \dim(W) = 2$, which proves independence of vectors $\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x$.

A

Pf. Choose $\vec{w} \in W$ where $\vec{w}: y = c_1 x + c_2 e^x$ for $c_1, c_2 \in \mathbb{R}$.
We show there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\begin{cases} c_1 = d_1 + d_2 \\ c_2 = d_1 - d_2 \end{cases} \qquad (1)$$

has a unique solution. We find

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$$

thus A^{-1} exists and the system (1) has a unique solution by the Invertible Matrix Theorem. Hence

$$\begin{aligned}\vec{w}: y &= (d_1+d_2)x + (d_1-d_2)e^x\\ &= d_1(x+e^x) + d_2(x-e^x)\\ &= d_1\vec{v}_1 + d_2\vec{v}_2\end{aligned}$$

and $\vec{w} \in S$. Therefore $S \subseteq W$ and $W \subseteq S$, meaning $S = W$.
It follows that $\dim S = \dim W = 2$, meaning $\vec{v}_1$ and $\vec{v}_2$ are independent. □

Problem 1. **(100 points)** Let V be the vector space of all functions on $(-\infty, \infty)$. Define $W = \mathbf{span}\{x, e^x\}$. Assume known that x, e^x are independent functions. Define subspace $S = \mathbf{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ where

$$\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x.$$

A

(a) [20%] Explain why S is contained in W, that is, provide details for why linear combinations of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are in W.

Let $\bar{s} \in S$. $\bar{s} = c_1(x+e^x) + c_2(x-e^x) = (c_1+c_2)x + (c_1-c_2)e^x$

Since $W = \text{span}\{x, e^x\} = d_1x + d_2e^x$, if the constants of $\bar{s}$ exist, $\bar{s} \in W \Rightarrow S \in W$.

$c_1 + c_2 = d_1$

$c_1 - c_2 = d_2$

(b) [40%] Prove that $W = S$. Therefore $\dim(S) = \dim(W) = 2$, which proves independence of vectors $\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x$.

A

Let $\bar{s} \in S$. From part (a), $\bar{s} \in W$.

Let $\bar{s} \in W \Rightarrow \bar{s} = d_1x + d_2e^x = c_1(x+e^x) + c_2(x-e^x) = (c_1+c_2)x + (c_1-c_2)e^x$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$, there are unique solutions for every combination of d_1 & d_2 $\Rightarrow \bar{s} \in W$

Since $\bar{s} \in W$ & $\bar{s} \in S \Rightarrow W = S$.

(c) [40%] Define vector $\vec{\mathbf{v}}$ in S by equation $y = 2x + 3e^x$. **Show how** to compute d_1, d_2 in the equation $\vec{\mathbf{v}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2$, **using coordinate map methods**. The definitions are $\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x$.

$\begin{bmatrix} x \\ e^x \end{bmatrix}$

A/A−

Expected in (c): Calculations of d_1, d_2 are to be done using column vectors from $\mathcal{R}^2$ and 2×2 matrices, not functions from V. **Zero credit** for not using column vectors $\boxed{\text{and coordinate maps.}}$

ok missing

$$\bar{v} = d_1\bar{v}_1 + d_2\bar{v}_2 = d_1(x+e^x) + d_2(x-e^x) = d_1\begin{bmatrix}1\\1\end{bmatrix} + d_2\begin{bmatrix}1\\-1\end{bmatrix}$$

$\bar{v} = 2x + 3e^x \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = 3 \end{matrix}$ Coordinate map used, but not explicitly.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1/2 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{bmatrix} \sim\!> \boxed{\begin{matrix} d_1 = \frac{5}{2} \\ d_2 = -\frac{1}{2} \end{matrix}}$$

missing:

$T: c_1 x + c_2 e^x \to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Apply T to eq. $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2$

$d_1 T(\vec{v}_1) + d_2 T(\vec{v}_2) = T(\vec{v})$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solve for $d_1 = \frac{5}{2}$, $d_2 = -\frac{1}{2}$ as above

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S

(2) If vectors $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x}+\vec{y}$ is in S.

(3) If vector $\vec{x}$ is in S and c is any scalar, then $c\vec{x}$ is in S.

Problem 2. (100 points)

A

(a) [60%] Let V be an *abstract* vector space. Let $\vec{v}_1, \vec{v}_2$ be two vectors in V. Define S to be the set of all linear combinations of $\vec{v}_1+\vec{v}_2, \vec{v}_1-\vec{v}_2$. Prove that S is a subspace of V, using only the definition of subspace.

Expected: A proof uses the symbols $\vec{v}_1, \vec{v}_2$ and the 8 rules of a vector space, plus theorems like $0\vec{v} = \vec{0}$. Symbols $\vec{v}_1, \vec{v}_2$ are not assumed to be column vectors.

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(1):

Prove $0 \in S$.

$S = \text{span}\{\bar{V}_1+\bar{V}_2, \bar{V}_1-\bar{V}_2\} = c_1(\bar{V}_1+\bar{V}_2) + c_2(\bar{V}_1-\bar{V}_2)$

If $c_1 = c_2 = 0$, $\Rightarrow S = \bar{0}$ ✓

(2) Prove $\bar{x}+\bar{y} \in S$.

Let $\bar{x} \in S$ & $\bar{y} \in S \Rightarrow \bar{x} = c_1(\bar{V}_1+\bar{V}_2) + c_2(\bar{V}_1-\bar{V}_2)$

$\bar{y} = d_1(\bar{V}_1+\bar{V}_2) + d_2(\bar{V}_1-\bar{V}_2)$

$\bar{x}+\bar{y} = c_1(\bar{V}_1+\bar{V}_2) + c_2(\bar{V}_1-\bar{V}_2) + d_1(\bar{V}_1+\bar{V}_2) + d_2(\bar{V}_1-\bar{V}_2)$

ing (1) from rules of a vector space:

$\bar{x}+\bar{y} = c_1(\bar{V}_1+\bar{V}_2) + d_1(\bar{V}_1+\bar{V}_2) + c_2(\bar{V}_1-\bar{V}_2) + d_2(\bar{V}_1-\bar{V}_2)$

sing (6) from rules of vector space:

$\bar{x}+\bar{y} = (c_1+d_1)(\bar{V}_1+\bar{V}_2) + (c_2+d_2)(\bar{V}_1-\bar{V}_2) \Rightarrow \bar{x}+\bar{y} \in S.$ ✓

3) Prove $a\bar{x} \in S$.

Let $x \in S. \Rightarrow \bar{x} = c_1(\bar{V}_1+\bar{V}_2) + c_2(\bar{V}_1-\bar{V}_2)$

Let $a \in \mathbb{R} \Rightarrow a\bar{x} = a[c_1(\bar{V}_1+\bar{V}_2) + c_2(V_1-V_2)]$

sing Rule (5):

$a\bar{x} = ac_1(V_1+V_2) + ac_2(V_1-V_2) \Rightarrow a\bar{x} \in S.$ ✓

By definition of a subspace, S is a subspace of V.

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S

(2) If vectors $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x}+\vec{y}$ is in S.

(3) If vector $\vec{x}$ is in S and c is any scalar, then $c\vec{x}$ is in S.

(b) [40%] Let V be the vector space of all 2×2 matrices. Invent an example of a non-void subset S of V that satisfies (1) and (2) but fails the third item (3).

A

Let $S = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$ s.t. $x \geq 0$ & $y \geq 0$.

(2) $\Rightarrow$ if $a_1 \in S$ & $a_2 \in S$ $\Rightarrow$ $a_1 + a_2 \in S$ since $x_1 + x_2 \geq 0$ & $y_1 + y_2 \geq 0$

(1) Additionally, $0 \in S$ since x can equal 0 & y can equal 0, so $\vec{0} \in S$

(3) However, if $c = -1$, $-\bar{x}$ is not in S, since $x \geq 0$ & $y \geq 0$, so it fails (3).

Problem 3. (100 points) Let A be a 4×3 matrix. Assume the determinant of $A^T A$ is zero. Prove that the nullspace of A contains a nonzero vector.

$|A^TA| = 0$ So A^TA is not invertible. By the invertible Matrix theorem, an invertible matrix must have a non-zero determinant.

A

So, A^TA has dependent cols.

$A^TA\vec{x} = \vec{0}$ has nontrivial soln

$$x^TA^TA\vec{x} = x^T\vec{0} \Rightarrow (A\vec{x})^TA\vec{x} = \vec{0} \Rightarrow \|A\vec{x}\|^2 = \vec{0} \Rightarrow \underbrace{A\vec{x} = \vec{0}}$$

has same relationship, has dependent cols b/c

Since $A\vec{x} = \vec{0}$ has sol where $\vec{x} \neq \vec{0}$, then the Nullspace has a nonzero vector $\leftarrow$ $\Big\{$ has sol where $\vec{x} \neq 0$

Problem 4. (100 points)

(a) [40%] Define $\vec{\mathbf{y}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Find the orthogonal projection vector $\vec{\mathbf{v}}$ (the shadow projection vector) of $\vec{\mathbf{y}}$ onto the direction of $\vec{\mathbf{u}}$.

A

$$\mathrm{Proj}_{\vec{u}}\vec{y} = \frac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{0}{2}\vec{u} = \boxed{\vec{0}}$$

(b) [60%] Let $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Check that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are orthogonal and then compute $\vec{\mathbf{v}} =$ the vector projection of $\vec{\mathbf{x}}$ onto the subspace $S = \mathbf{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$.

A

Reminder: $\vec{\mathbf{v}}$ is the sum of two shadow projections.

$$\vec{V}_1 \cdot \vec{V}_2 = 1 + {-1} + 0 = \boxed{0} \quad \checkmark \text{ orthogonal}$$

$$\mathrm{Proj}_{v_1}\vec{x} = \frac{\vec{v}_1\cdot\vec{x}}{\vec{V}_1\cdot\vec{V}_1}\vec{v}_1 = \frac{1+1+0}{1+1+0}\vec{v}_1 = 1\vec{v}_1 = \vec{v}_1$$

$$\mathrm{Proj}_{v_2}\vec{x} = \frac{\vec{V}_2\cdot\vec{x}}{V_2\cdot V_2}V_2 = \frac{1+{-1}+2}{1+1+1}V_2 = \frac{2}{3}V_2$$

$$\vec{V} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix} = \boxed{\begin{pmatrix} 5/3 \\ 1/3 \\ 2/3 \end{pmatrix}}$$

Problem 5. (100 points) Let A be an $m \times n$ matrix and $\vec{\mathbf{b}}$ an $m \times 1$ vector. Let W be the column space of A. Linear equations $A^T A\vec{\mathbf{z}} = A^T\vec{\mathbf{b}}$ are the **normal equations** for the problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(a) [30%] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Display toolkit steps that verify $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution.

$A\vec{x} = \vec{b} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 0 & -1 \end{array}\right]$ combo(1,2,-1)

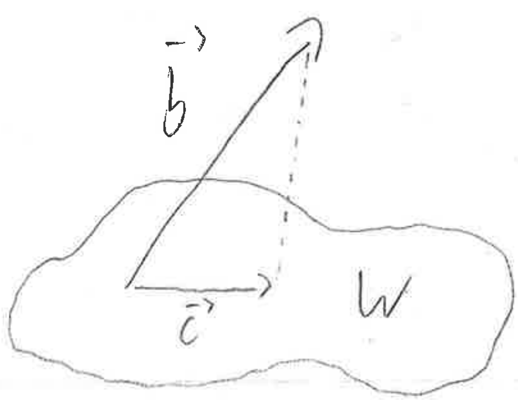
A

$0x_1 + 0x_2 = -2$ is a signal equation

$\Rightarrow A\vec{x} = \vec{b}$ has no solution

(b) [30%] Let $\vec{\mathbf{c}} = A\vec{\mathbf{z}}$ where $\vec{\mathbf{z}}$ is the unique theoretical solution of the normal equations. Explain with a figure: $\vec{\mathbf{c}}$ is the nearest point to $\vec{\mathbf{b}}$ in the column space W.

A



(c) [40%] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Find vector $\vec{\mathbf{z}}$ in the normal equations.

Use $A^T A \vec{z} = A^T \vec{b}$ to create normal eq and solve for $\vec{z}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{z} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \vec{z} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{aligned} 3x_1 + 2x_2 &= -1 \\ 2x_1 + 2x_2 &= 0 \end{aligned}$$

Solve:

$$\left[\begin{array}{cc|c} 3 & 2 & -1 \\ 2 & 2 & 0 \end{array}\right] \overset{\text{combo}(2,1,-1)}{\approx} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 2 & 2 & 0 \end{array}\right] \overset{\text{mult}(2, \frac{1}{2})}{\approx} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 1 & 1 & 0 \end{array}\right]$$

$$\overset{\text{combo}(1,2,-1)}{\approx} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= -1 \\ x_2 &= 1 \end{aligned} \qquad \vec{z} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Problem 6. **(100 points)** The **Fundamental Theorem of Linear Algebra** contains this statement: The row space and the null space of a matrix are orthogonal. This means that $\vec{\mathbf{R}} \cdot \vec{\mathbf{N}} = 0$ for each vector $\vec{\mathbf{R}}$ in the row space and each vector $\vec{\mathbf{N}}$ in the null space.

The **four fundamental subspaces** in the Fundamental Theorem of Linear algebra are: (1) Nullspace of A, (2) Column Space of A, (3) Row space of A, 4) Nullspace of A^T.

(a) [30%] Define precisely the four fundamental subspaces. For example, the Nullspace of A is the set of all solutions $\vec{\mathbf{x}}$ to the matrix equation $A\vec{\mathbf{x}} = \vec{0}$.

A

(1) The Nullspace of A is the set of all solutions to the homogeneous eq. $A\vec{x} = \vec{0}$.

(2) The Column Space of A is the set of all l.c. of the columns of A.

(3) The Row space of A is the set of all l.c. of the rows of A (equal to the col. space of A^T)

(4) The Nullspace of A^T is the set of all nonzero rows of A

(b) [30%] Assume A is 20×12 and has rank 10. Equivalently, matrix A has 10 pivots . Report the dimensions of the four fundamental subspaces.

A

If $m = 20$ $n = 12$ $r = 10$

dim Nullspace $(A) = n - r = 12 - 10 = \boxed{2}$

dim Col$(A) = r = \boxed{10}$

dim Row$(A) = r = \boxed{10}$

dim Nullspace$(A^T) = \boxed{10}$

if A has 10 pivot col

A^T has 10 pivot rows

A has 10

non zero rows

(c) [40%] Let A be an $m \times n$ matrix. Let $\vec{\mathbf{C}}$ be a linear combination of the columns of A and let $\vec{\mathbf{Y}}$ belong to the nullspace of A^T. Prove that $\vec{\mathbf{C}} \cdot \vec{\mathbf{Y}} = 0$, that is, $\vec{\mathbf{C}}$ and $\vec{\mathbf{Y}}$ are orthogonal.

The Fundamental Theorem of ~~Al~~ gebra states that

A

Row (A) $\perp$ Nullspace (A)

Similarly, Row$(A^T) \perp$ Nullspace(A^T)

$\vec{C}$ = part of Col (A)   By definition

$\vec{Y}$ = part of Null (A^T) of A^T, Col (A) = Row(A^T)

Therefore, $\vec{C}$ = part of Row(A^T)

and by Fund. Theorem of L.A.,

$\vec{C} \cdot \vec{Y} = \vec{0}$ b/c Row$(A^T) \perp$ Null(A^T)

and $\vec{C}$ and $\vec{Y}$ are orthogonal.

(c) [40%] Let A be an $m \times n$ matrix. Let $\vec{\mathbf{C}}$ be a linear combination of the columns of A and let $\vec{\mathbf{Y}}$ belong to the nullspace of A^T. Prove that $\vec{\mathbf{C}} \cdot \vec{\mathbf{Y}} = 0$, that is, $\vec{\mathbf{C}}$ and $\vec{\mathbf{Y}}$ are orthogonal.

A. $\vec{C} \cdot \vec{Y} = A\vec{x} \cdot \vec{y}$ ($\vec{x}$ being the solution to $A\vec{x} = \vec{c}$, known to exist by the definition of linear combination)

$A\vec{x} \cdot \vec{y} = (A\vec{x})^T \vec{y} = \vec{x}^T A^T \vec{y}$

$A^T \vec{y} = \vec{0}$, by the definition of nullspace

$\vec{x}^T A^T \vec{y} = \vec{x}^T \vec{0} = 0$

Thus, $\vec{C}$ and $\vec{y}$ are orthogonal

Excellent