MATH 2270-4 Exam 3 S2019

NAME: ______________________________

Please, no books, notes or electronic devices.

Some questions involve proofs. Please divide your time accordingly.

Extra details can appear on the back side or on extra pages. Please supply a road map for details not directly following the problem statement.

Details count 75% and answers count 25%.

QUESTION	VALUE	SCORE
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
TOTAL	600	

Definition: An *abstract* vector space V is a data set of packages called **vectors** together with operations of addition ($+$) and scalar multiplication ($\cdot$) satisfying the following eight (8) rules:

Closure: If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in V, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is defined and in V.

(1) $\vec{\mathbf{x}} + \vec{\mathbf{y}} = \vec{\mathbf{y}} + \vec{\mathbf{x}}$

(2) $\vec{\mathbf{x}} + (\vec{\mathbf{y}} + \vec{\mathbf{z}}) = (\vec{\mathbf{x}} + \vec{\mathbf{y}}) + \vec{\mathbf{z}}$

(3) There is a zero vector $\vec{\mathbf{0}}$ in V with $\vec{\mathbf{x}} + \vec{\mathbf{0}} = \vec{\mathbf{x}}$.

(4) There is a vector $-\vec{\mathbf{x}}$ in V with $\vec{\mathbf{x}} + (-\vec{\mathbf{x}}) = \vec{\mathbf{0}}$.

Closure: If $c =$ constant and $\vec{\mathbf{x}}$ is in V, then $c \cdot \vec{\mathbf{x}}$ is defined and in V.

(5) $a(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = a\vec{\mathbf{x}} + a\vec{\mathbf{y}}$

(6) $(a + b) \cdot \vec{\mathbf{x}} = a \cdot \vec{\mathbf{x}} + b \cdot \vec{\mathbf{x}}$

(7) $(ab) \cdot \vec{\mathbf{x}} = a \cdot (b \cdot \vec{\mathbf{x}})$

(8) $1 \cdot \vec{\mathbf{x}} = \vec{\mathbf{x}}$

Definition. If vectors $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n$ are a basis for subspace X of an *abstract* vector space V, and $\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + \cdots + c_n\vec{\mathbf{b}}_n$ is a given linear combination of these vectors, then the uniquely determined constants $c_1, c_2, \ldots, c_n$ are called the *coordinates of* $\vec{\mathbf{x}}$ *relative to the basis* $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n$ and the *coordinate map* is the isomorphism

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + \cdots + c_n\vec{\mathbf{b}}_n \to \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S

(2) If vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is in S.

(3) If vector $\vec{\mathbf{x}}$ is in S and c is any scalar, then $c\vec{\mathbf{x}}$ is in S.

Definition: Vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ in an abstract vector space V are said to be **independent** in V provided solving the equation $c_1\vec{\mathbf{v}}_1 + \cdots + c_p\vec{\mathbf{v}}_p = \vec{\mathbf{0}}$ for scalars $c_1, \ldots, c_p$ has only the zero solution $c_1 = \cdots = c_p = 0$.

Problem 1. (100 points) Let V be the vector space of all functions on $(-\infty, \infty)$. Define $W = \mathbf{span}\{x, e^x\}$. Assume known that x, e^x are independent functions. Define subspace $S = \mathbf{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ where

$$\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x.$$

(a) [20%] Explain why S is contained in W, that is, provide details for why linear combinations of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are in W.

(b) [40%] Prove that $W = S$. Therefore $\dim(S) = \dim(W) = 2$, which proves independence of vectors $\vec{\mathbf{v}}_1 : \ y = x + e^x, \quad \vec{\mathbf{v}}_2 : \ y = x - e^x$.

(c) [40%] Define vector $\vec{\mathbf{v}}$ in S by equation $y = 2x + 3e^x$. **Show how** to compute d_1, d_2 in the equation $\vec{\mathbf{v}} = d_1\vec{\mathbf{v}}_1 + d_2\vec{\mathbf{v}}_2$, **using coordinate map methods**. The definitions are $\vec{\mathbf{v}}_1 : \; y = x + e^x, \quad \vec{\mathbf{v}}_2 : \; y = x - e^x$.

Expected in (c): Calculations of d_1, d_2 are to be done using column vectors from $\mathcal{R}^2$ and 2×2 matrices, not functions from V. **Zero credit** for not using column vectors and coordinate maps.

Definition: A subset S of a vector space V is a **subspace** of V provided

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Problem 2. (100 points)

(a) [60%] Let V be an *abstract* vector space. Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ be two vectors in V. Define S to be the set of all linear combinations of $\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$. Prove that S is a subspace of V, using only the definition of subspace.

> **Expected**: A proof uses the symbols $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ and the 8 rules of a vector space, plus theorems like $0\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Symbols $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are not assumed to be column vectors.

Definition: A subset S of a vector space V is a **subspace** of V provided

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(2) If vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is in S.

(3) If vector $\vec{\mathbf{x}}$ is in S and c is any scalar, then $c\vec{\mathbf{x}}$ is in S.

(b) [40%] Let V be the vector space of all 2×2 matrices. Invent an example of a non-void subset S of V that satisfies (1) and (2) but fails the third item (3).

Problem 3. (100 points) Let A be a 4×3 matrix. Assume the determinant of $A^T A$ is zero. Prove that the nullspace of A contains a nonzero vector.

Problem 4. (100 points)

(a) [40%] Define $\vec{\mathbf{y}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Find the orthogonal projection vector $\vec{\mathbf{v}}$ (the shadow projection vector) of $\vec{\mathbf{y}}$ onto the direction of $\vec{\mathbf{u}}$.

(b) [60%] Let $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Check that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are orthogonal and then compute $\vec{\mathbf{v}}$ = the vector projection of $\vec{\mathbf{x}}$ onto the subspace $S = \mathbf{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$.

Reminder: $\vec{\mathbf{v}}$ is the sum of two shadow projections.

Problem 5. (100 points) Let A be an $m \times n$ matrix and $\vec{\mathbf{b}}$ an $m \times 1$ vector. Let W be the column space of A. Linear equations $A^T A\vec{\mathbf{z}} = A^T\vec{\mathbf{b}}$ are the **normal equations** for the problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(a) [30%] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Display toolkit steps that verify $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution.

(b) [30%] Let $\vec{\mathbf{c}} = A\vec{\mathbf{z}}$ where $\vec{\mathbf{z}}$ is the unique theoretical solution of the normal equations. Explain with a figure: $\vec{\mathbf{c}}$ is the nearest point to $\vec{\mathbf{b}}$ in the column space W.

(c) [40%] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Find vector $\vec{\mathbf{z}}$ in the normal equations.

Problem 6. (100 points) The **Fundamental Theorem of Linear Algebra** contains this statement: The row space and the null space of a matrix are orthogonal. This means that $\vec{\mathbf{R}} \cdot \vec{\mathbf{N}} = 0$ for each vector $\vec{\mathbf{R}}$ in the row space and each vector $\vec{\mathbf{N}}$ in the null space.

The **four fundamental subspaces** in the Fundamental Theorem of Linear algebra are: (1) Nullspace of A, (2) Column Space of A, (3) Row space of A, 4) Nullspace of A^T.

(a) [30%] Define precisely the four fundamental subspaces. For example, the Nullspace of A is the set of all solutions $\vec{\mathbf{x}}$ to the matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(b) [30%] Assume A is 20×12 and has rank 10. Equivalently, matrix A has 10 pivots . Report the dimensions of the four fundamental subspaces.

(c) [40%] Let A be an $m \times n$ matrix. Let $\vec{\mathbf{C}}$ be a linear combination of the columns of A and let $\vec{\mathbf{Y}}$ belong to the nullspace of A^T. Prove that $\vec{\mathbf{C}} \cdot \vec{\mathbf{Y}} = 0$, that is, $\vec{\mathbf{C}}$ and $\vec{\mathbf{Y}}$ are orthogonal.