MATH 2270-2 Final Exam Spring 2016

1. (Chapter 1: 60 points) Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ defined by

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 &= 2 \\ 4x_1 + 3x_2 + 8x_3 + x_4 &= 4 \\ 6x_1 + 3x_2 + 8x_3 + x_4 &= 2 \end{aligned}$$

Solve the following parts:

(a) [10%] Find the reduced row echelon form of the augmented matrix.

(b) [10%] Identify the **free** variables and the **lead** variables.

(c) [10%] Display a vector formula for a particular solution $\vec{u}_p$.

(d) [10%] Display a vector formula for the homogeneous solution $\vec{u}_h$.

(e) [10%] Identify each of **Strang's Special Solutions**.

(f) [10%] Display the vector general solution $\vec{u}$, using **superposition**.

2. (Chapter 2: 40 points)
(a) [10%] Describe for $n \times n$ matrices two different methods for finding the matrix inverse.
(b) [20%] Apply the two methods to find the inverse of the matrix $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$.
(c) [10%] Find the inverse of the transpose of the matrix in part (b).

3. (Chapter 3: 30 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

4. (Chapters 1 to 4: 30 points) Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ -1 & 0 & 0 \\ 6 & 6 & 3 \\ 2 & 2 & 1 \end{pmatrix}$$

(a) Check the independence tests below which apply to prove that the column vectors of the matrix A are independent in the vector space $\mathcal{R}^4$.
(b) Show the details for one of the independence tests that you checked.

☐	**Wronskian test**	Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Euler Atom test**	Any finite set of distinct atoms is independent.
☐	**Sample test**	Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Orthogonality test**	A set of nonzero pairwise orthogonal vectors is independent.
☐	**Combination test**	A list of vectors is independent if each vector is not a linear combination of the preceding vectors.

5. (Chapters 2, 4: 20 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

6. (Chapter 6: 40 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S.

7. (Chapters 1 to 6: 30 points) Let A be an $m \times n$ matrix and assume that $A^T A$ has nonzero determinant. Prove that the rank of A equals n.

8. (Chapter 5: 40 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display three eigenpairs of A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

9. (Chapter 6: 30 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\hat{\vec{\mathbf{b}}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\hat{\vec{\mathbf{b}}}$.

10. (Chapter 6: 30 points) Let Q be an orthogonal matrix with columns $\vec{q}_1, \vec{q}_2, \vec{q}_3$. Let D be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$. Prove that the 3×3 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T}$.

11. (Chapter 7: 30 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

12. (Chapter 7: 30 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

13. (Chapter 4: 30 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors:

$$T\left(\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}, T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

14. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. It is known that S_1 and S_2 have dimension

$r = \textbf{rank}(A)$. Let $\vec{p}_1, \ldots, \vec{p}_r$ be a basis for S_1 and let $\vec{q}_1, \ldots, \vec{q}_r$ be a basis for S_2. For example, select the pivot columns of A^T and A, respectively. Define $T : S_1 \to S_2$ initially by $T(\vec{p}_i) = \vec{q}_i, \quad i = 1, \ldots, r$. Extend T to all of S_1 by linearity, which means the final definition is

$$T(c_1\vec{p}_1 + \cdots + c_r\vec{p}_r) = c_1\vec{q}_1 + \cdots + c_r\vec{q}_r.$$

Prove that T is one-to-one and onto.

15. (Chapter 4: 20 points) Least squares can be used to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Without finding the line equation, describe how to do it, in a few sentences.

16. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

17. (Chapter 7: 20 points) State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition**.