MATH 2270-2 Final Exam Spring 2018

Problem 1. (Chapter 1: 100 points) Solve Linear Algebraic Equations.

Solve the system $A\vec{\mathbf{u}} = \vec{\mathbf{b}}$ defined by

$$\begin{cases} 2x_1 + x_2 + 8x_3 + x_4 + 2x_5 &= 4 \\ x_1 + 3x_2 + 4x_3 + x_4 + x_5 &= 2 \\ 2x_1 + 2x_2 + 8x_3 + x_4 + x_5 &= 4 \end{cases} \qquad \vec{\mathbf{u}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \vec{\mathbf{b}} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}.$$

Expected: (a) Augmented matrix and Toolkit steps to the reduced row echelon form (RREF). (b) Translation of RREF to scalar equations. Identify lead and free variables. (c) Scalar general solution for x_1 to x_5. (d) Write vector formulas for the homogeneous solution $\vec{\mathbf{u}}_h$, a particular solution $\vec{\mathbf{u}}_p$ and the general solution $\vec{\mathbf{u}} = \vec{\mathbf{u}}_h + \vec{\mathbf{u}}_p$.

Problem 2. (Chapters 2 and 3: 100 points) Inverse, Determinant, Cramer's Rule.

(a) [40%] Find the inverse of the matrix $A = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) [30%] Let A be defined as in part (a). Compute the determinant of $((A + A^T)^{-1})^T$.

(c) [30%] Let P, Q, R denote undisclosed real numbers. Define matrix B and vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{c}}$ by the equations

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{\mathbf{c}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $B\vec{\mathbf{x}} = \vec{\mathbf{c}}$.

Theorem (Wronskian test). Wronskian determinant of f_1, f_2, f_3 nonzero at some invented $x = x_0$ implies independence of f_1, f_2, f_3.
Theorem (Sampling test). Functions f_1, f_2, f_3 are independent if a sampling matrix constructed for some invented samples x_1, x_2, x_3 has nonzero determinant.

Problem 3. (Chapters 1 to 4: 100 points) Independence Tests for Functions.

Let V be the vector space of all functions on $(-\infty, \infty)$. Define functions in V by the equations $f_1(x) = 5 + e^x$, $f_2(x) = 2x$, $f_3(x) = x + x^2$.

(a) [50%] Construct the Wronskian matrix W of the given functions f_1, f_2, f_3, then invent a value for x such that $|W| \neq 0$.

(b) [50%] Construct a sampling matrix S for the given functions f_1, f_2, f_3, using invented samples x_1, x_2, x_3, such that $|S| \neq 0$.

Rank test. Vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent if their augmented matrix has rank 3.

Determinant test. Vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent if their square augmented matrix has nonzero determinant.

Pivot test. Vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent if their augmented matrix A has 3 pivot columns.

Orthogonality test. A set of nonzero pairwise orthogonal vectors is independent.

Problem 4. (Chapters 1 to 4: 100 points) Independence Tests for Column Vectors.

Let V be the vector space of all functions on $(-\infty, \infty)$. It is known that the functions $g_1(x) = 5 + e^x$, $g_2(x) = 2x - e^x$, $g_3(x) = e^x$ are independent in V. Let $S = \mathbf{span}(g_1, g_2, g_3)$. Define a coordinate map isomorphism from S to $\mathcal{R}^3$ by

$$T \;:\; c_1(5 + e^x) + c_2(2x - e^x) + c_3(e^x) \quad \text{maps into} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

(a) [60%] Define functions in V by the equations $f_1(x) = 5 + e^x$, $f_2(x) = 5 + 2x$, $f_3(x) = 2x$. Determine the column vectors $\vec{\mathbf{v}}_1 = T(f_1), \vec{\mathbf{v}}_2 = T(f_2), \vec{\mathbf{v}}_3 = T(f_3)$.

(b) [40%] Because T is one-to-one and onto, then the given functions f_1, f_2, f_3 are independent in S if and only if the column vectors $T(f_1), T(f_2), T(f_3)$ are independent in $\mathcal{R}^3$. Show details for **one** of the above independence tests applied to the column vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ calculated in part (a) above.

DEFINITION. Subset S of vector space V is a subspace of V provided (1), (2), (3) hold:
(1) S contains vector $\vec{\mathbf{0}}$.
(2) If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is in S.
(3) If c is a constant and $\vec{\mathbf{x}}$ is in S, then $c\vec{\mathbf{x}}$ is in S.

Problem 5. (Chapters 2, 4 and 6: 100 points) Subspace Definition and Theorems.

(a) [50%] Set S consists of all vectors $\vec{\mathbf{x}}$ with components x_1, x_2, x_3 in vector space $\mathcal{R}^3$ such that $x_1 + 2x_3 = x_2$ and $x_1 + x_3 = 0$. Supply proof details which verify that S is a subspace of $\mathcal{R}^3$. Cite a theorem or else verify conditions (1), (2), (3) in the DEFINITION.

(b) [50%] Set S consists of all vectors $\vec{\mathbf{x}}$ in vector space $\mathcal{R}^4$ which are linear combinations of the vectors

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 7 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 8 \end{pmatrix}.$$

Supply proof details which verify that S is a subspace of $\mathcal{R}^4$. Cite a theorem or else verify conditions (1), (2), (3) in the DEFINITION.

Problem 6. (Chapter 6: 100 points) Gram-Schmidt Orthogonalization Process.

Let S be the subspace of $\mathcal{R}^4$ spanned by the independent vectors

$$\vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthogonal basis $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ for subspace S.

Problem 7. (Chapters 1 to 6: 100 points) Symmetric Matrices and the Invertible Matrix Theorem.

Let A be an $m \times n$ matrix and assume that $A^T A$ is invertible. Prove that the columns of A are linearly independent.

Expected: A referenced result from *"The Invertible Matrix Theorem"* should appear as a precisely stated LEMMA, the proof of the LEMMA deferred to the textbook.

Problem 8. (Chapter 5: 100 points) Eigenanalysis and Diagonalization.

The matrix A below has eigenvalues 2, 8 and 8.

$$A = \begin{pmatrix} 12 & 4 & -1 \\ -4 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) [80%] Compute all eigenpairs of A.

Expected: For each eigenvalue λ, first compute the RREF of $A - \lambda I$, then compute all eigenvectors for λ (they are Strang's solutions).

(b) [20%] Is A diagonalizable? Explain why or why not.

Problem 9. (Chapter 6: 100 points) Near Point Theorem and Least Squares.

(a) [60%] Define $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Write the normal equations for the inconsistent problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and solve for the least squares solution $\hat{\mathbf{x}}$.

(b) [20%] Least squares can be used to find the best fit line $y = mx + b$ for the (x, y)-data points $(-1, 3)$, $(0, 1)$, $(1, 2)$. Find the line equation by the method of least squares.

Expected: The matrix A you create for part (b) should match the matrix A of part (a). Save time by using the computations from (a).

(c) [20%] Continue part (a). Compute vector $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$, which is the near point to $\vec{\mathbf{b}}$ in the column space of A. Then compute the mean square error, which is the norm of the vector $\vec{\mathbf{b}} - \hat{\mathbf{b}}$.

Problem 10. (Chapter 7: 100 points) Spectral Theorem and $AQ = QD$.

The spectral theorem says that a symmetric matrix A satisfies $AQ = QD$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$.

Problem 11. (Chapter 7: 100 points) Singular Value Decomposition.

Determine the singular value decomposition for the matrix $A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$.

Problem 12. (Chapter 4: 100 points) Linear Transformations as Matrix Multiply.

Let the linear transformation T from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

$$T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Find the unique 2×2 matrix A such that T is defined by the matrix multiply equation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

Problem 13. (Chapters 4 and 6: 100 points) Orthogonality.

Let symbols a, b, c, d, e, f represent certain real numbers. Define $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Define subspaces

$$S_1 = \text{the column space of the transpose matrix } A^T = \mathbf{Col}(A^T)$$
$$S_2 = \text{the null space of } A = \mathbf{Null}(A).$$

Let $\vec{\mathbf{x}}$ belong to S_1 and let $\vec{\mathbf{y}}$ belong to S_2. Prove that their dot product is zero: $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = 0$.

Expected: Apply the definition of matrix multiply in terms of dot products. No theorems are used, only definitions.

Problem 14. (Chapters 1 to 7: 100 points) Fundamental Theorem of Linear Algebra.

(a) [40%] Give a technical definition for each of the four fundamental subspaces appearing in the Fundamental Theorem of Linear Algebra. With each definition, describe how to compute a basis for the subspace.

(b) [30%] What are the dimensions of the four subspaces?

(c) [30%] State the orthogonality relations for the four fundamental subspaces.
