MATH 2270-4 Sample Exam 2 S2019

No books or notes. No electronic devices, please.

These problems have credits 5 to 20, which is an estimate of the time required to write the solution.

This sample will be edited during the semester to reflect course content. Expect to see problems that appeared on the 2270 final exam in 2018.

The number of problems on Exam 2 is eight (8). This sample has more than 30 problems, in order to illustrate possible problem types. See the 2016, 2017 and 2018 Exam2 (PDF) questions and answers for a typical exam.

Problem 1. (10 points) Determinant problem, chapter 3.
Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & \frac{1}{2} \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 4, column 2 of A^{-1}.

Problem 2. (10 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

Problem 3. (10 points) Assume given 3×3 matrices A, B. Suppose $E_3E_2E_1A = BA^2$ and E_1, E_2, E_3 are elementary matrices representing respectively a multiply by 3, a swap and a combination. Assume $\det(B) = 3$. Find all possible values of $\det(-2A)$.

Problem 4. (5 points) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Show the details of two different methods for finding A^{-1}.

Problem 5. (10 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Problem 6. (10 points)
Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

Problem 7. (10 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

Problem 8. (10 points) Using the subspace criterion, write two hypotheses that imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Problem 9. (10 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 10. (10 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Problem 11. (5 points) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Problem 12. (5 points) A 10×13 matrix A is given and the homogeneous system

$A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

Problem 13. (5 points) The rank of a 10×13 matrix A is 7. Find the nullity of A.

Problem 14. (10 points) Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ of $\mathcal{R}^2$, and $\vec{v} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of $\vec{v}$ relative to the basis $\vec{v}_1, \vec{v}_2$*. Compute c_1 and c_2.

Problem 15. (10 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Problem 16. (10 points) Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sampling test**	Sample matrix of f_1, f_2, f_3 at samples $x = x_1, x_2, x_3$ with nonzero determinant implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Problem 17. (10 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Problem 18. (10 points) The 5×6 matrix A below has some independent columns.

Report a maximum number of independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Problem 19. (10 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

Problem 20. (10 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

Problem 21. (10 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Problem 22. (15 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Problem 23. (10 points) The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$. To save time, **don't find the eigenvector for** $\lambda = 0$.

Problem 24. (10 points) Find the two eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Problem 25. (10 points) Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Problem 26. (10 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Problem 27. (10 points) Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Problem 28. (10 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Problem 29. (10 points) Assume A is a given 4×4 matrix with eigenvalues 0, 1, $3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

Problem 30. (20 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Problem 31. (10 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [50%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [50%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

Problem 32. (10 points) Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

Problem 33. (10 points) Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

Problem 34. (10 points) Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

No new questions beyond this point. Please check back at the course web site until 22 March, for corrections and added sample exam problems.