MATH 2270-2 Exam 2 S2016

NAME: ____________________

Please, no books, notes or electronic devices.

The last four (4) questions are proofs. Please divide your time accordingly.

Extra details can be the back side or on extra pages. Please supply a road map for details not on the front side.

Details count 75% and answers count 25%.

QUESTION	VALUE	SCORE
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
7	100	
8	100	
9	100	
10	100	
11	100	
12	100	
TOTAL	1200	

Problem 1. (100 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_3 by Cramer's Rule, showing **all details** (details count 75%). To save time, **do not compute** x_1, x_2!

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Problem 3. (100 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

Please display answers for both $\vec{x}_h$ and $\vec{x}_p$. The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Problem 4. (100 points) Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$ for a subspace S of $\mathcal{R}^3$, and $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ in S, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of* $\vec{v}$ *relative to the basis* $\vec{v}_1, \vec{v}_2$. Compute c_1 and c_2.

Problem 5. (100 points) The functions 1, x^2, $\sqrt{x^7}$ are independent in the vector space V of all functions on $0 < x < \infty$. Check the independence tests which apply.

☐	**Wronskian test**	Wronskian determinant of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sampling test**	Sampling determinant nonzero for samples $x = x_1$, x_2, x_3 implies independence of f_1, f_2, f_3.
☐	**Rank test**	Three vectors are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Three vectors are independent if their augmented matrix is square and has nonzero determinant.
☐	**Orthogonality test**	Three vectors are independent if they are all nonzero and pairwise orthogonal.
☐	**Pivot test**	Three vectors are independent if their augmented matrix A has 3 pivot columns.

Problem 6. **(100 points)** Consider a 3×3 real matrix A with eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}\right), \quad \left(1+i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1-i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right).$$

(a) [60%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(b) [40%] Display a matrix product formula for A. To save time, **do not evaluate any matrix products**.

Problem 7. **(100 points)** The matrix $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ -1 & -1 & 4 \end{pmatrix}$ has eigenvalues $2, 4, 4$.

Find all eigenvectors for $\lambda = 4$. To save time, **don't find the eigenvector for** $\lambda = 2$. Then report whether or not matrix A is diagonalizable.

Problem 8. (100 points) Using the subspace criterion, write three different hypotheses each of which imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Problem 9. (100 points) Define S to be the set of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$, $x_3 + x_2 = x_1$ and $x_1 - x_3 = 0$. Prove that S is a subspace of $\mathcal{R}^3$.

Problem 10. **(100 points)** Let A be a 100×29 matrix. Assume the columns of $A^T A$ are independent. Prove that A is invertible.

Problem 11. (100 points) Let 3×3 matrices A, B and C be related by $AP = PB$ and $BQ = QC$ for some invertible matrices P and Q. Prove that the characteristic equations of A and C are identical.

Problem 12. (100 points) The **Fundamental Theorem of Linear Algebra** says that the null space of a matrix is orthogonal to the row space of the matrix.

Let A be an $m \times n$ matrix. Define subspaces S_1 = column space of A, S_2 = null space of A^T. Prove that a vector $\vec{v}$ orthogonal to S_2 must be in S_1.