MATH 2270-2 Exam 2 S2018

NAME: ____________________

Please, no books, notes or electronic devices.

Questions 4, 8, 9, 10 involve proofs. Please divide your time accordingly.

Extra details can appear on the back side or on extra pages. Please supply a road map for details not on the front side.

Details count 75% and answers count 25%.

QUESTION	VALUE	SCORE
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
7	100	
8	100	
9	100	
10	100	
TOTAL	1000	

Problem 1. **(100 points)** Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, display the **formula** for x_3 according to Cramer's Rule. To save time, **do not compute determinants**!

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Problem 3. (100 points) Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Expected: (a) Augmented matrix. (b) Toolkit steps to RREF. (c) Translation of RREF to scalar equations. (d) Scalar general solution. (e) Find the homogeneous solution $\vec{x}_h$, which is a linear combination of Strang's special solutions. (f) Find a particular solution $\vec{x}_p$. (g) Write the vector general solution $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$.

Problem 4. (100 points)

> **Definition**. If $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ are a basis for subspace W of vector space V, and $\vec{\mathbf{x}} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ is a given linear combination of these vectors, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of $\vec{x}$ relative to the basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$*.

Below, let V be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \mathbf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = 1 + x$, $y = 2 + x^2$, $y = 2 + x + x^2$.

(a) [40%] Let $W = \mathbf{span}\{1, x, x^2\}$. Assume known that $1, x, x^2$ are independent functions. Already, $S = \mathbf{span}\{1 + x, 2 + x^2, 2 + x + x^2\}$ is a subset of W. **Prove** that W is a subset of S (this proves that $W = S$, therefore $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent).

(b) [60%] Define vector $\vec{\mathbf{v}}$ in S by equation $y = 3 + 4x + x^2$. **Compute** c_1, c_2, c_3 satisfying the equation $\vec{v} = c_1\vec{\mathbf{v}}1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$, using coordinate map methods.

Expected in (b): Vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are defined by $1+x, 2+x^2, 2+x+x^2$, respectively. Calculations of c_1, c_2, c_3 are to be done using column vectors from $\mathcal{R}^3$, not functions from V. **Zero credit** for not using column vectors.

Problem 5. **(100 points)** The functions 1, x^2, x^4 represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space V of all functions on $0 < x < \infty$. The set $S = \textbf{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a subspace of V. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in V be defined by the functions $1-x^2, x^4+x^2, 3+2x^4$, respectively. The **coordinate map** defined by

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathcal{R}^3$, respectively:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space V or in vector space $\mathcal{R}^3$, because the coordinate map takes independent sets to independent sets.

Check below all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Explain how each checked test applies, giving details/reasons. **Zero credit for checking a box without explanation**.

☐ **Wronskian test** Nonzero Wronskian determinant of f_1, f_2, f_3 at invented value $x = x_0$ implies independence of f_1, f_2, f_3.

Explain:

☐ **Sampling test** Nonzero sampling determinant for invented samples $x = x_1, x_2, x_3$ implies independence of f_1, f_2, f_3.

Explain:

☐ **Rank test** Three column vectors are independent if their augmented matrix has rank 3.

Explain:

☐ **Determinant test** Three column vectors are independent if their augmented matrix is square and has nonzero determinant.

Explain:

☐ **Orthogonality test** Three column vectors are independent if they are all nonzero and pairwise orthogonal.

Explain:

☐ **Pivot test** Three column vectors are independent if their augmented matrix A has 3 pivot columns.

Explain:

Problem 6. **(100 points)** Consider a 3×3 real matrix A with eigenpairs

$$\left(5, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right), \quad \left(2+i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2-i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right).$$

(a) [30%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$. Matrices P and D can contain complex numbers.

(b) [30%] Display a **real** invertible matrix P_1 and a **real** diagonal matrix D_1 such that $AP_1 = P_1 D_1$. Neither P_1 nor D_1 can contain complex numbers. The construction of D_1 uses the map $a + ib \rightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

(c) [40%] Display a matrix product formula for A in which the factors contain only real numbers. To save time, **do not evaluate any matrix products**.

Problem 7. (100 points)

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S

(2) If vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is in S.

(3) If vector $\vec{\mathbf{x}}$ is in S and c is any scalar, then $c\vec{\mathbf{x}}$ is in S.

Let V be the vector space of all real-valued functions on $(-\infty, \infty)$. Invent an example of a nonvoid subset S of V that satisfies two of the items in the above definition of subspace, but fails the third item.

Problem 8. (100 points) Define S to be the set of all vectors $\vec{\mathbf{x}}$ in $\mathcal{R}^3$ whose components x_1, x_2, x_3 satisfy the two restriction equations $x_1 + x_2 = x_3$ and $2x_1 + 5x_2 = x_3$. Prove that S is a subspace of $\mathcal{R}^3$.

Expected: Cite a known theorem or else verify the 3 conditions for the definition of a subspace (see the preceding exam problem).

Problem 9. (100 points) Let A be a 4×3 matrix. Assume the columns of $A^T A$ are independent. Prove or disprove that A has independent columns.

Expected: To prove a claim, assemble details and theorem citations to support the claim. To disprove a claim, invent a specific detailed example that violates the claim.

Problem 10. (100 points) Let U be a 2×2 matrix with $U^T U = I$. Let $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ denote the columns of U. Prove that the columns of U are orthonormal.