MATH 2270-2 Exam 2 S2018

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Please, no books, notes or electronic devices.

Questions 4, 8, 9, 10 involve proofs. Please divide your time accordingly.

Extra details can appear on the back side or on extra pages. Please supply a road map for details not on the front side.

Details count 75% and answers count 25%.

QUESTION	VALUE	SCORE
A 1	100	
A 2	100	
A- 3	100	-7
A I A 4	100	
BBBB A- B 5	100	-8 -1 -2
A B A 6	100	-5
A 7	100	
A 8	100	
A 9	100	
A- 10	100	-6
TOTAL	1000	-29 = 971

Problem 1. (100 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, display the formula for x_3 according to Cramer's Rule. To save time, **do not compute determinants!**

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & -3 \\ 0 & -4 & 5 \\ 1 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{vmatrix}}$$

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

$$\begin{bmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

Problem 3. **(100 points)** Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

A-

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Expected: (a) Augmented matrix. (b) Toolkit steps to RREF. (c) Translation of RREF to scalar equations. (d) Scalar general solution. (e) Find the homogeneous solution $\vec{x}_h$, which is a linear combination of Strang's special solutions. (f) Find a particular solution $\vec{x}_p$. (g) Write the vector general solution $\vec{x} = \vec{x}_h + \vec{x}_p$.

$$\left[\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 3 & 3 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left(\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 0 & 3 & 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 4/3 \\ 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 & \text{free} \\ x_2 = 4/3 + x_5 \\ x_3 = -1 - x_5 \\ x_4 & \text{free} \\ x_5 & \text{free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_5 + \frac{4}{3} \\ -x_5 - 1 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 4/3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = \underbrace{t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ 4/3 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p}$$

Problem 4. (100 points)

Definition. If $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are a basis for subspace W of vector space V, and $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ is a given linear combination of these vectors, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of $\vec{x}$ relative to the basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$.*

Below, let V be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = 1 + x$, $y = 2 + x^2$, $y = 2 + x + x^2$.

A (a) [40%] Let $W = \text{span}\{1, x, x^2\}$. Assume known that $1, x, x^2$ are independent functions. Already, $S = \text{span}\{1 + x, 2 + x^2, 2 + x + x^2\}$ is a subset of W. Prove that W is a subset of S (this proves that $W = S$, therefore $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent).

A (b) [60%] Define vector $\vec{v}$ in S by equation $y = 3 + 4x + x^2$. Compute c_1, c_2, c_3 satisfying the equation $\vec{v} = c_1\vec{v}1 + c_2\vec{v}_2 + c_3\vec{v}_3$, using coordinate map methods.

Expected in (b): Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are defined by $1+x, 2+x^2, 2+x+x^2$, respectively. Calculations of c_1, c_2, c_3 are to be done using column vectors from $\mathcal{R}^3$, not functions from V. **Zero credit** for not using column vectors.

(a) Let $\vec{v} \in W$, then $\vec{v}$ is of the form

$c_1 + c_2 x + c_3 x^2$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Have that any $\vec{s} \in S$ is of the form $c_1(1+x) + c_2(2+x^2) + c_3(2+x+x^2)$

$= c_1 + c_1 x + 2c_2 + c_2 x^2 + 2c_3 + c_3 x + c_3 x^2 = (c_1 + 2c_2 + 2c_3) + (c_1 + c_3)x + (c_2 + c_3)x^2$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Therefore, $\vec{v} \in S$ and $W \subseteq S$.

(b) $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$[\vec{v}]_W = c_1[\vec{v}_1]_W + c_2[\vec{v}_2]_W + c_3[\vec{v}_3]_W$

$[\vec{v}]_W = \left[[\vec{v}_1]_W \;\; [\vec{v}_2]_W \;\; [\vec{v}_3]_W \right] (\vec{v})_c$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, the coordinates c_1, c_2, c_3 are

$$\begin{cases} c_1 & 1 \\ c_2 & -2 \\ c_3 = 3 \end{cases}$$

Problem 5. **(100 points)** The functions 1, x^2, x^4 represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space V of all functions on $0 < x < \infty$. The set $S = \mathbf{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a subspace of V. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in V be defined by the functions $1-x^2$, x^4+x^2, $3+2x^4$, respectively. The **coordinate map** defined by

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathcal{R}^3$, respectively:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space V or in vector space $\mathcal{R}^3$, because the coordinate map takes independent sets to independent sets.

Check below all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Explain how each checked test applies, giving details/reasons. **Zero credit for checking a box without explanation.**

☑ **Wronskian test** Nonzero Wronskian determinant of f_1, f_2, f_3 at invented value $x = x_0$ implies independence of f_1, f_2, f_3.

B

Explain: $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are functions, so you can test independence directly with this method.

☑ **Sampling test** Nonzero sampling determinant for invented samples $x = x_1, x_2, x_3$ implies independence of f_1, f_2, f_3.

B

Explain: Like above, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are functions, let $\vec{v}_1 = f_1$, $\vec{v}_2 = f_2$, $\vec{v}_3 = f_3$ and can be teste directly with this method.

☑ **Rank test** Three column vectors are independent if their augmented matrix has rank 3.

B

Explain: $\vec{v}_1, \vec{v}_2, \vec{v}_3$ can be mapped to column vectors in $\mathbb{R}^3$ by the coordinate mapping.

☑ **Determinant test** Three column vectors are independent if their augmented matrix is square and has nonzero determinant.

B

Explain: Same as above, the coordinate mapping is isomorphic and will persene the dependence relation between the vectors

☐ **Orthogonality test** Three column vectors are independent if they are all nonzero and pairwise orthogonal.

A-

Explain: Expected: Test $v_1 \cdot v_2 = 0$ for col vectors v_1, v_2.

☑ **Pivot test** Three column vectors are independent if their augmented matrix A has 3 pivot columns.

B

Explain: Same as above, this is actually equivalent to the rank test.

Expected: Display matrix A, toolkit sequence, RREF, determinant.
For Wronskian: Display matrix W, choose $x = x_0$ to make $|W| \neq 0$
" Sampling: Display matrix, select samples x_1, x_2, x_3 then Test $\det \neq 0$.

Problem 6. (100 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(5, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right), \quad \left(2+i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2-i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right).$$

A **(a)** [30%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$. Matrices P and D can contain complex numbers.

B **(b)** [30%] Display a real invertible matrix P_1 and a real diagonal matrix D_1 such that $AP_1 = P_1 D_1$. Neither P_1 nor D_1 can contain complex numbers. The construction of D_1 uses the map $a + ib \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

A **(c)** [40%] Display a matrix product formula for A in which the factors contain only real numbers. To save time, do not evaluate any matrix products.

(a) A has 3 linearly independent eigenvector, so by the diagonalization theorem

$AP = PD$ where $P = \begin{bmatrix} 1 & i & -i \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$

(B) $D_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ Has to be 3×3

(c) Since $AP_1 = P_1 D \Rightarrow A = P_1 D P_1^{-1}$

Problem 7. (100 points)

Definition: A subset S of a vector space V is a subspace of V provided

(1) The zero vector is in S

(2) If vectors $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x}+\vec{y}$ is in S.

(3) If vector $\vec{x}$ is in S and c is any scalar, then $c\vec{x}$ is in S.

A

Let V be the vector space of all real-valued functions on $(-\infty, \infty)$. Invent an example of a nonvoid subset S of V that satisfies two of the items in the above definition of subspace, but fails the third item.

Let S be the set of functions defined on $[0, \infty]$ and where f is ~~positive~~. NON-Negative

then $\vec{0} \in S$, $\vec{x}+\vec{y} \in f$, but $c\vec{x}$ is not in

because $0 \geq 0$

S, because c can be negative

$-\infty$

$$f : [0, \infty] \longrightarrow [0, \infty]$$

Need $f \in V$

~~$f(x) = x^a$ for all $a \in \mathbb{R}$~~

~~$\sin x$, $\cos x$~~ ~~there is no a where $f(x) = 0$.~~

~~So zero vector not in set.~~

~~$x^a + x^a = 2x^a$~~

Problem 8. (100 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ whose components x_1, x_2, x_3 satisfy the two restriction equations $x_1 + x_2 = x_3$ and $2x_1 + 5x_2 = x_3$. Prove that S is a subspace of $\mathcal{R}^3$.

Expected: Cite a known theorem or else verify the 3 conditions for the definition of a subspace (see the preceding exam problem).

A

Have that $\begin{array}{l} x_1 + x_2 = x_3 \\ 2x_1 + 5x_2 = x_3 \end{array} \implies x_1 + 4x_2 = 0 \implies \begin{array}{l} x_2 = -x_1/4 \\ x_3 = \frac{3x_1}{4} \end{array}$

So the set is all vectors of the form

$$\vec{v} = \begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} \text{ for all } t \in \mathbb{R}. = t\left(\begin{array}{c}\text{Strang's} \\ \text{Sol}\end{array}\right)$$

Use Span Theorem: $S = \text{Span}\left(\begin{array}{c}\text{strang's} \\ \text{Sol}\end{array}\right)$ is a subspace.

Let $t = 0$, then

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the zero vector is in S.

Let $\vec{v}$ and $\vec{w}$ be arbitrary vectors in S, then

$$\vec{v} + \vec{w} = \begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} + \begin{bmatrix} s \\ -s/4 \\ 3s/4 \end{bmatrix} = \begin{bmatrix} s+t \\ -\frac{(s+t)}{4} \\ \frac{3(s+t)}{4} \end{bmatrix}, \text{ so } \vec{v} + \vec{w} \in S$$

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}$, then

$$c\vec{v} = c\begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} = \begin{bmatrix} ct \\ -(ct)/4 \\ 3(ct)/4 \end{bmatrix} \text{ so } c\vec{v} \in S.$$

Therefore, S is a subspace of $\mathbb{R}^n$.

Problem 9. (100 points) Let A be a 4×3 matrix. Assume the columns of $A^T A$ are independent. Prove or disprove that A has independent columns.

Expected: To prove a claim, assemble details and theorem citations to support the claim. To disprove a claim, invent a specific detailed example that violates the claim.

A

Assume for contradiction that A has dependent columns. Then, there is a nontrivial solution to $A\vec{x} = \vec{0}$.

However,

$$A\vec{x} = \vec{0} \iff A^T A \vec{x} = A^T \vec{0} \iff A^T A \vec{x} = \vec{0}$$

Therefore, $A^T A \vec{x} = \vec{0}$ shares this nontrivial solution.

But this a contradiction, since we assumed the columns of $A^T A$ are independent.

Therefore, A must have independent columns.

Problem 10. (100 points) Let U be a 2×2 matrix with $U^T U = I$. Let $\vec{u}_1, \vec{u}_2$ denote the columns of U. Prove that the columns of U are orthonormal.

A-

$$U^T U = \begin{bmatrix} U^T \vec{u}_1 & U^T \vec{u}_2 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}$$

or

$$\boxed{\begin{bmatrix} u_1^T u_1 \\ u_2^T u_1 \end{bmatrix}} = U^T \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad U^T \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \|u_1\|^2 = 1$ & $u_2 \cdot u_1 = 0$ $\qquad \Rightarrow \|u_2\|^2 = 1$ & $u_1 \cdot u_2 = 0$

~~Since $U^T U = I$, U is invertible and U^T is the inverse of U.~~ true

~~Then, the columns of U are linearly independent~~ true

~~and $U \sim I$.~~ true ~~Therefore,~~

~~$\vec{u}_1 \cdot \vec{u}_2 = 0$~~ leap of faith...

Extraneous material that does not show

$\|Col_1\| = \|Col_2\| = 1$

&

$Col_1 \perp Col_2$