MATH 2270-2 Exam 2 S2017

Please, no books, notes or electronic devices.

The last four (4) questions are proofs. Please divide your time accordingly.

Extra details can be on the back side or on extra pages. Please supply a road map for details not on the front side.

Details count 75% and answers count 25%.

Problem 1. (100 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} z_1 & z_2 & 0 \\ 0 & z_3 & 0 \\ 1 & z_4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, display the formula for x_2 according to Cramer's Rule. To save time, **do not compute determinants**!

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & 1 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Problem 3. (100 points) Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Please display vector answers for both $\vec{x}_h$ and $\vec{x}_p$. The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Problem 4. (100 points) Let V be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \mathbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = x - 1$, $y = 1 + x^2$, $y = 2x + x^2$. If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of $\vec{v}$ relative to the basis* $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Compute c_1, c_2, c_3 for $\vec{v}$ defined by $y = 1 + 2x + 3x^2$

Problem 5. (100 points) The functions 1, x^2, x^5 represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space V of all functions on $0 < x < \infty$. The set $S = \textbf{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a subspace of V. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in V be defined by the functions $1 + x^2, x^5 - x^2, 5 + 2x^5$, respectively. The **coordinate map** defined by

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathcal{R}^3$, respectively:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}.$$

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space V or in vector space $\mathcal{R}^3$, because the coordinate map takes independent sets to independent sets.

Check below all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

☐	**Wronskian test**	Wronskian determinant of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sampling test**	Sampling determinant for samples $x = x_1, x_2, x_3$ nonzero implies independence of f_1, f_2, f_3.
☐	**Rank test**	Three vectors are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Three vectors are independent if their augmented matrix is square and has nonzero determinant.
☐	**Orthogonality test**	Three vectors are independent if they are all nonzero and pairwise orthogonal.
☐	**Pivot test**	Three vectors are independent if their augmented matrix A has 3 pivot columns.

Problem 6. (100 points) The matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ -1 & -1 & 5 \end{pmatrix}$ has eigenvalues $3, 4, 4$.

(a) [80%] Find all eigenvectors for $\lambda = 4$. To save time, **don't find** $\lambda = 3$ **eigenvectors**.

(b) [20%] Report whether or not matrix A is diagonalizable. Explain.

Problem 7. (100 points) Define S to be the set of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$ and $x_3 x_2 = x_1 x_2$. Show that S is **NOT** a subspace of $\mathcal{R}^3$, that is, exhibit a counterexample to one of the items in the *Subspace Criterion*.

Problem 8. (100 points) Let A be a 4×3 matrix. Assume the columns of $A^T A$ are dependent. Prove or disprove that A has dependent columns.

Problem 9. (100 points) Let 3×3 matrices A, B and C be related by $AP = PB$ and $BQ = QC$ for some invertible matrices P and Q. Assume B has eigenvalues $2, 3, 7$. Prove that matrices A and C also have eigenvalues $2, 3, 7$.

Problem 10. (100 points) The **Fundamental Theorem of Linear Algebra** says that the null space of a matrix is orthogonal to the row space of the matrix.

Let A be an $m \times n$ matrix. Define subspaces S_1 = column space of A, S_2 = null space of A^T. Prove that the only vector $\vec{v}$ in both S_1 and S_2 is the zero vector.