Problem 1. (100 points) Define matrix $A$, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix}
-2 & 3 & 0 & 0 \\
-4 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ -5 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

(a) [40%] For the system $A\vec{x} = \vec{b}$, display the formula for $x_2$ according to Cramer's Rule. Don't compute $x_2$! Don't expand determinants!

$$x_2 = \frac{|A_{2}(\vec{b})|}{|A|} = \frac{\begin{vmatrix}
-2 & 3 & 0 & 0 \\
0 & -5 & 0 & 0 \\
1 & 4 & 1 & 0 \\
0 & 0 & 0 & 2
\end{vmatrix}}{16}.$$

(b) [60%] Find the entry in row 3 and column 2 in matrix $A^{-1}$, by using the adjugate formula for the inverse: $A^{-1} = \frac{\text{adj}(A)}{|A|}$.

The answer is a fraction. Matrix $A$ is not triangular, but cofactor expansion applies: $|A| = 16$.

$$\text{adj}(A_{3,2}) = \text{cofactor}(A_{2,3}) = -\begin{vmatrix}
-2 & 3 & 0 \\
0 & -5 & 0 \\
1 & 4 & 1 \\
0 & 0 & 2
\end{vmatrix} \quad \text{along row 3: } - (2) \begin{vmatrix}
0 & 0 \\
1 & 4 \\
0 & 2
\end{vmatrix} = - (2) (-11) = 22.$$
Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix}$. Find a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.

\[
A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix} \sim E_1 \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 14 & -4 \end{pmatrix} \sim E_2 \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -4 \end{pmatrix} = U \quad \& \quad U = E_3 E_2 E_1 A
\]

\[
L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}
L = E_1^{-1} E_2^{-1} E_3^{-1}
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}
E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Check: $LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix} \quad \checkmark$
Problem 3. (100 points) Vector space \( V \) is the set of all functions on \( 0 < x < \infty \).
Equations \( y = 1, y = x^2, y = x^3 \) represent independent vectors \( \vec{b}_1, \vec{b}_2, \vec{b}_3 \) in \( V \) and \( S = \text{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \) is a subspace of \( V \). The coordinate map \( T \) from \( S \) to \( \mathbb{R}^3 \) is defined by

\[
c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3 \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \text{ or } c_1 + c_2 x^2 + c_3 x^3 \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.
\]

Define vectors in subspace \( S \):

\[
\vec{v}_1 : \quad y = 1 - x^2, \quad \vec{v}_2 : \quad y = x^3 - x^2, \quad \vec{v}_3 : \quad y = 4 + 2x^3.
\]

Vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are mapped by \( T \) as follows:

\[
1 - x^2 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x^3 - x^2 \rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad 4 + 2x^3 \rightarrow \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.
\]

The coordinate map \( T \), an isomorphism, maps independent sets to independent sets. Therefore, the set \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) is independent in \( V \) if and only if the three column vectors above are independent in \( \mathbb{R}^3 \).

Apply each of the three independence tests below to establish independence of \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \). Details are expected: explain briefly how the test applies. Zero credit for no explanation.

The phrase augmented matrix used below means the \( 3 \times 3 \) matrix \( (\vec{v}_1 | \vec{v}_2 | \vec{v}_3) \).
Problem 4 Continued.

\[ \text{Wronskian test. Nonzero Wronskian determinant of } f_1, f_2, f_3 \text{ at invented value } x = x_0 \implies \text{independence of } f_1, f_2, f_3. \]

Details: applied to functions
\[
W(x) = \begin{vmatrix}
1 \cdot x^2 & x^2 - x^2 & 4 + 2x^2 \\
-2x & 3x^2 - 2x & 6x^2 \\
-2 & 6x - 2 & 12x
\end{vmatrix} \implies |W(1)| = \begin{vmatrix}
0 & 0 & 6 \\
-2 & 1 & 0 \\
-2 & 4 & 12
\end{vmatrix} = 6(-8 + 2) = -36 \neq 0
\]

\[ \text{independent: } c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = 0 \quad (\text{where } \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are functions}). \text{To solve this function, make a system of eqns of derivatives, } c_1 \vec{v}_1' + c_2 \vec{v}_2' + c_3 \vec{v}_3' = 0 \quad \text{&} \quad c_1 \vec{v}_1'' + c_2 \vec{v}_2'' + c_3 \vec{v}_3'' = 0. \text{ To solve for } c_1, c_2, c_3, \text{ if the determinant is not 0, it has an inverse } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \text{ Meaning the vectors are independent.} \]

Determinant test. Three column vectors are independent if their augmented matrix is square and has nonzero determinant.

Details: applied to fixed vectors
\[
\begin{pmatrix} 1 & 0 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{cof. expansion along column } 1 \quad |1, 2| = 1(-2 - 4) = -6 \neq 0 \quad \text{independent}
\]

If \[ |\begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{vmatrix}| \neq 0, \text{ the matrix is invertible } c_i \text{ by invertible matrix theorem, the columns of } [\vec{v}_1, \vec{v}_2, \vec{v}_3] \text{ are linearly independent.} \]

A Pivot test. Three column vectors are independent if their augmented matrix \( A \) has 3 pivot columns.

Details: if rref has 3 pivot columns \( \implies c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = 0 \) iff \( c_1 = 0, c_2 = 0, c_3 = 0 \)

which shows the 3 vectors are linearly independent.

\[
A = \begin{bmatrix}
1 & 0 & 4 \\
-1 & -1 & 0 \\
0 & 1 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 2 \\
0 & -1 & -4
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 4 \\
0 & 0 & 6
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 4 \\
0 & 0 & 6
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

3 pivot columns \( \implies \) independent.
Problem 4. (100 points) Matrix $A = \begin{pmatrix} 0 & 4 & -1 \\ 4 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ has real eigenpairs $(5, \left( -1 \middle\mid 1 \right))$, $(4, \left( 1 \middle\mid 1 \right))$, $(4, \left( -1 \middle\mid 1 \right))$.

(a) [30%] Display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.

\[ P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix} \]

(b) [20%] Display a symbolic matrix product formula for $A$ in terms of $P$ and $D$. To save time, do not evaluate anything.

\[ AP = PD \text{ where } P \text{ is invertible, so} \]

(c) [50%] Show the details for computing an eigenvector for $\lambda = 5$. \( \gamma (A - \lambda I) x = 0 \)

\[ \begin{align*}
\lambda &= 5 \\
\left[ A - \lambda I \right] &= \begin{pmatrix} -5 & 4 & -1 \\ 4 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -4/5 & 1/5 \\ 4 & -5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -4/5 & 1/5 \\ 0 & -9/5 & 1/5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -4/5 & 1/5 \\ 0 & 1 & -1/9 \\ 0 & 0 & 0 \end{pmatrix} \\
\sim \begin{pmatrix} 1 & 0 & 1/9 \\ 0 & 1 & -1/9 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 1/9 x_3 = 0 \\ x_2 - 1/9 x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -1/9 t_1 \\ x_2 = 1/9 t_1 \end{cases} \Rightarrow \tilde{x} = t_1 \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} \quad \text{eigenvector} \end{align*} \]

\[ (5, \left( -1 \middle\mid 1 \right)) \]
Problem 5. (100 points)

Definition: A subset $S$ of a vector space $V$ is a subspace of $V$ provided

1. The zero vector is in $S$.
2. If vectors $x$ and $y$ are in $S$, then $x + y$ is in $S$.
3. If vector $x$ is in $S$ and $c$ is any scalar, then $cx$ is in $S$.

Let vector space $V = \mathbb{R}^n$ and let $A$ be a given $m \times n$ matrix.

(a) [60%] Prove by definition that the equation $Ax = \bar{0}$ defines a subspace $S$ of $V$.

Let $\bar{u} \in S \Rightarrow A\bar{u} = \bar{0}$, let $\bar{v} \in S \Rightarrow A\bar{v} = \bar{0}$

$\bar{0}$ is contained in $S$ because $A(\bar{0}) = \bar{0}$

If $\bar{u}$ & $\bar{v}$ are in $S$ (defined above), $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = \bar{0} + \bar{0} = \bar{0}$

so $\bar{u} + \bar{v} \in S$.

If $\bar{u}$ is in $S$ (defined above), $A(c\bar{u}) = cA\bar{u} = c(\bar{0}) = 0$, for any constant $c \in \mathbb{R}$.

so $c\bar{u} \in S$, so $S$ is a subspace of $V$.

(b) [40%] Explain why the equation $Ax = \bar{b}$ fails to define a subspace of $V$ when $\bar{b} \neq \bar{0}$.

If $\bar{b} \neq \bar{0}$, let $A\bar{u} = \bar{b}$, so $\bar{u}$ is in the subset.

$\bar{0}$ is not in the subset because $A(\bar{0}) = \bar{0}$, but $\bar{b} \neq \bar{0}$

Since $\bar{0}$ is not in $S$, $S$ is not a subspace. Fails to meet definition of subspace.

(Not a Subspace Thm)
Problem 6. (100 points) Let \( \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \). Define \( S \) to be the set of all vectors \( \vec{x} \) in \( \mathbb{R}^3 \) which satisfy the two restriction equations \( \vec{v}_1 \cdot \vec{x} = 0, \vec{v}_2 \cdot \vec{x} = 0 \). Prove that \( S \) is a subspace of \( \mathbb{R}^3 \).

Expected: Cite known theorems, if they apply, to avoid writing a proof. If no theorems are applied, then verify the 3 conditions for the definition of a subspace (see the preceding exam problem).

Let \( \vec{u} \in S \implies \vec{v}_1 \cdot \vec{u} = 0, \vec{v}_2 \cdot \vec{u} = 0 \). Let \( \vec{y} \in S \). \( \vec{v}_1 \cdot \vec{y} = 0, \vec{v}_2 \cdot \vec{y} = 0 \)

- \( \vec{0} \) is contained in the subset \( S \) because \( \vec{v}_1 \cdot \vec{0} = 0 \) and \( \vec{v}_2 \cdot \vec{0} = 0 \)
- \( \vec{u} + \vec{y} \) are in the set \( S \). \( \vec{v}_1 \cdot (\vec{u} + \vec{y}) = \vec{v}_1 \cdot \vec{u} + \vec{v}_1 \cdot \vec{y} = 0 + 0 = 0 \)
- \( \vec{v}_2 \cdot (\vec{u} + \vec{y}) = \vec{v}_2 \cdot \vec{u} + \vec{v}_2 \cdot \vec{y} = 0 + 0 = 0 \), so \( \vec{u} + \vec{y} \) is in \( S \).
- \( \vec{u} \) is in \( S \), and \( \vec{v}_1 \cdot (c \vec{u}) = c (\vec{v}_1 \cdot \vec{u}) = c(0) = 0 \) for any constant \( c \), so \( c \vec{u} \) is in \( S \).

\( \therefore \) by definition of a subspace, \( S \) is a subspace of \( \mathbb{R}^3 \).
Problem 7. (100 points) Used in this problem are equivalent statements taken from the Invertible Matrix Theorem, which says that a square matrix $C$ has an inverse $C^{-1}$ if and only if one of the statements labeled $a$ to $x$ is true. Three of these statements, for example, are (1) $|C| \neq 0$, (2) $C$ has independent columns, (3) the dimension of the nullspace of $C$ is zero.

(a) [20%] Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Compute the $3 \times 3$ matrix $A^T A$.

\[
A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

(b) [80%] Let matrix $B$ be $2 \times 3$ with dependent columns. Prove or disprove: The $3 \times 3$ matrix $B^T B$ has dependent columns.

Expected: To prove a claim, assemble details and theorem citations to support the claim. To disprove a claim, invent a specific detailed example that violates the claim.

Assume $B^T B$ has independent columns. $B$ has dependent columns, so $Bx = 0$ has a nontrivial solution. Multiplying both sides by $B^T$, $B^T (Bx) = B^T (0) = (B^T B) x = 0$. Because $B^T B$ has independent columns, it is invertible by the Invertible Matrix Theorem (2), so $(B^T B)^{-1} (B^T B) x = (B^T B)^{-1} 0$.

\[Ix = 0 \Rightarrow x = 0.\] However $x$ cannot be the trivial solution since $Bx = 0$ has a nontrivial solution.

Thereby, by contradiction, $B^T B$ must have dependent columns.