Problem 1. (100 points) Matrix Algebra, Chapters 1,2.
Symbol I is used below for the $n \times n$ identity. Notation C^T means the transpose of matrix C. Accept as known theorems the following results:

Theorem 1. If C and D are $n \times n$ and $CD = I$, then $DC = I$.
Theorem 2. If A and B are invertible $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
Theorem 3. If matrices F, G have dimensions allowing FG to be defined, then $(FG)^T = G^T F^T$.
Theorem 4. If C is $n \times n$ invertible, then C^T is invertible and $(C^T)^{-1} = (C^{-1})^T$. A Excellent

In the statement below, either invent a counter example or else explain why it is true (citing relevant theorems above). Used in the theorems is the definition of inverse: G has an inverse H if and only if $GH = I$ and $HG = I$.

If matrices A, B are $n \times n$ with $A^T A = I$, then A^{-1} exists and $A^{-1}(A + B^T) = I + (BA)^T$. $(A^T A)^{-1} = I^{-1}$

If $A^T A = I$, by Theorem 1, $AA^T = I$. By definition of inverses, A^T has an inverse, so $AA^{-1} = I = A^{-1}A$, $\Rightarrow$ $A^{-1} = A^T$ ✓

If A^{-1} exists, $A^{-1}(A+B^T) = A^{-1}A + A^{-1}B^T$ by matrix mult. properties

$A^{-1}A + A^{-1}B^T = I + A^{-1}B^T$ by property of identity.

$I + A^{-1}B^T = I + A^T B^T$ by substitution of $A^T = A^{-1}$

$I + A^T B^T = I + (BA)^T$ by Theorem 3.

$\Rightarrow A^{-1}(A+B^T) = I + (BA)^T$

TRUE

Problem 2. **(100 points)** Elementary Matrices and Toolkit Sequences, Chapters 1,2.

> **Definition**: An **elementary matrix** E is the matrix answer after applying exactly one combo, swap or multiply to the identity matrix I. An **elimination matrix** M is a product of elementary matrices.

Let A be a 3×4 matrix. Find the elimination matrix M which under left multiplication against matrix A performs (1), (2) and (3) below with one matrix multiply.

$MA \Rightarrow (3\times3)(3\times4) = 3\times4$ $\qquad$ M is 3x3 matrix

(1) Replace Row 3 of A with Row 3 minus twice Row 2 to obtain new matrix A_1.

(2) Swap Row 1 and Row 3 of A_1 to obtain new matrix A_2.

(3) Multiply Row 3 of A_2 by 1/5 to obtain new matrix A_3.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1) \quad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$(2) \quad A_2 = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(3) \quad A_3 = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1/5 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1/5 & 0 & 0 \end{bmatrix}$$

Problem 3. (100 points) Linear algebraic equations.
System $A\vec{u} = \vec{b}$ with symbols. The Three Possibilities. Chapters 1,2,3.

Let symbols a, b and c denote constants and consider the system of equations

$$\begin{cases} x + by + cz = a \\ 2x + (b+c)y - az = -a \\ x + cy + az = -a \end{cases}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

A (a) [40%] The system has a unique solution for $(c-b)(2a+b) \neq 0$.

$2ac + bc - 2ab - b^2$

A (b) [30%] The system has no solution if $2a + c = 0$ and $a \neq 0$ (don't explain the other possibilities).

A- (c) [30%] The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities). Explain no signal eq.

cite a Theorem. Details 75%

a) If solution is unique of $Ax=b \Rightarrow x = A^{-1}b \Rightarrow A^{-1}$ must exist
For A^{-1} to exist, $|A| \neq 0$

$$A = \begin{bmatrix} 1 & b & c \\ 2 & b+c & -a \\ 1 & c & a \end{bmatrix}$$

(cofactor expansion on Row 1)

$$|A| = +1\left[(b+c)a + ac\right] - b\left[2a + a\right] + c\left[2c - (b+c)\right]$$
$$= ab + ac + ac - 2ab - ab + 2c^2 - bc - c^2$$
$$|A| = -2ab + 2ac + c^2 - bc$$
$$= 2a(-b+c) + c(c-b) = (c-b)(2a+c)$$

If $(c-b)(2a+b) \neq 0 \Rightarrow |A| \neq 0$, so A has an inverse and a unique solution exists.

b)
$$\left[\begin{array}{ccc|c} 1 & b & c & a \\ 2 & b+c & -a & -a \\ 1 & c & a & -a \end{array}\right] \xrightarrow[\text{combo}(1,3,-1)]{\text{combo}(1,2,-2)} \left[\begin{array}{ccc|c} 1 & b & c & a \\ 0 & -b+c & -a-2c & -3a \\ 0 & -b+c & a-c & -2a \end{array}\right] \xrightarrow{\text{combo}(2,3,-1)} \left[\begin{array}{ccc|c} 1 & b & c & a \\ 0 & -b+c & -a-2c & -3a \\ 0 & 0 & 2a+c & a \end{array}\right]$$

$2a + c = a$ by last row of matrix
If $2a+c=0$ & $a \neq 0 \Rightarrow 0 = a$ where $a \neq 0$, so this is a signal equation & the system has no solution.

$\longrightarrow$

c) $2a + c = a$ by last row in reduced matrix.

If $a = b = c = 0 \Rightarrow 0 = 0$ is last row of reduced matrix

$\Rightarrow z$ is a free variable so there are infinitely many solutions.

Definition. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided solving vector equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ results in the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called **dependent**.

Problem 4. (100 points) Linear Independence, Chapters 1,2,3.
Solve parts (a), (b) and (c) using the vectors displayed below. Application of theorems is expected: the Pivot Theorem, the Rank Test, the Determinant Test. Or, directly use the definition of independence (above). Details are 75%, answer 25%.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

A **(a)** [50%] Show details for the dependence of the 4 vectors.
A **(b)** [20%] List a maximum number of independent vectors extracted from the 4 vectors.
A **(c)** [30%] Write each vector not listed in **(b)** as a linear combination of the reported independent vectors.

*Thm: If one vector is the 0 vector $\Rightarrow$ linearly dependent vectors

a) $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 + c_4\bar{v}_4 = 0$

pivot thm:

$$\begin{bmatrix} 0&0&1&1 \\ 2&0&2&4 \\ 2&0&1&3 \\ 0&0&2&2 \end{bmatrix} \overset{\text{swap}(1,2)}{\Rightarrow} \begin{bmatrix} 2&0&2&4 \\ 0&0&1&1 \\ 2&0&1&3 \\ 0&0&2&2 \end{bmatrix} \overset{\text{combo}(1,3,-1)}{\Rightarrow} \begin{bmatrix} 2&0&2&4 \\ 0&0&1&1 \\ 0&0&-1&-1 \\ 0&0&2&2 \end{bmatrix} \overset{\text{mult}(1,1/2)}{\Rightarrow}$$

$$\begin{bmatrix} 1&0&1&2 \\ 0&0&1&1 \\ 0&0&-1&-1 \\ 0&0&2&2 \end{bmatrix} \overset{\text{combo}(2,3,1)}{\Rightarrow} \begin{bmatrix} 1&0&1&2 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&2&2 \end{bmatrix} \overset{\text{combo}(2,4,-2)}{\Rightarrow} \begin{bmatrix} 1&0&1&2 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \text{ combo}(2,1,-1)$$

$$\begin{bmatrix} 1&0&0&1 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \text{ RREF}$$

By pivot Theorem, only 2 columns of the 4 are pivot columns so the 4 vectors are dependent.

b) only columns 1 & 3 are pivot columns $\Rightarrow$ max independent vectors = 2

$\boxed{\vec{V}_1 \ \& \ \vec{V}_3}$

$\longrightarrow$

c) $c_1\vec{V_1} + c_3\vec{V_3} = \vec{V_2}$

$$c_1\begin{bmatrix}0\\2\\2\\0\end{bmatrix} + c_3\begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad \left.\begin{array}{r}c_3=0\\2c_1+2c_3=0\\2c_1+c_3=0\\2c_3=0\end{array}\right\} \begin{array}{l}c_1=0\\c_3=0\end{array}$$

$$\boxed{\vec{V_2} = 0\vec{V_1} + 0\vec{V_3}} \Rightarrow \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 0\begin{pmatrix}0\\2\\2\\0\end{pmatrix} + 0\begin{pmatrix}1\\2\\1\\2\end{pmatrix}$$

$c_1\vec{V_1} + c_3\vec{V_3} = \vec{V_4}$

$$c_1\begin{bmatrix}0\\2\\2\\0\end{bmatrix} + c_3\begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\4\\3\\2\end{bmatrix} \qquad \left.\begin{array}{r}c_3=1\\2c_1+2c_3=4\\2c_1+c_3=3\\2c_3=2\end{array}\right\} \begin{array}{l}c_1=1\\c_3=1\end{array}$$

$$\boxed{\vec{V_4} = 1\vec{V_1} + 1\vec{V_3}} \Rightarrow \begin{pmatrix}1\\4\\3\\2\end{pmatrix} = 1\begin{pmatrix}0\\2\\2\\0\end{pmatrix} + 1\begin{pmatrix}1\\2\\1\\2\end{pmatrix}$$

Problem 5. (100 points) Vector general solution of a matrix equation $A\vec{x} = \vec{b}$, Chapters 1,2.

Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

A

$$A = \begin{pmatrix} 0 & 1 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 5 \\ 3 \\ 8 \\ 0 \end{pmatrix}$$

Expected: (a) [10%] Augmented matrix, (b) [40%] Toolkit steps for the RREF, (c) [10%] Conversion of RREF to scalar equations, (d) [20%] Last frame Algorithm details to write out the scalar general solution, (e) [20%] Conversion of the scalar general solution to the vector general solution. This answer is in the form of a single vector equation for $\vec{x}$, the solution of system $A\vec{x} = \vec{b}$. The expected components of $\vec{x}$ are x_1, x_2, x_3, x_4.

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 4 & 5 \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 4 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow{\text{combo}(1,2,-3)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & -12 & -12 \\ 0 & 4 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow{\text{combo}(1,3,-4)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & -12 & -12 \\ 0 & 0 & 1 & -12 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xRightarrow{\text{combo}(2,3,-1)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & -12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

*2 lead variables
2 free variables

$x_2 + 4x_4 = 5$
$x_3 - 12x_4 = -12$
$0 = 0$
$0 = 0$

$x_1 = \text{free}$
$x_2 = 5 - 4x_4$
$x_3 = -12 + 12x_4$
$x_4 = \text{free}$

$\Rightarrow$

$x_1 = t_1$
$x_2 = 5 - 4t_2$
$x_3 = -12 + 12t_2$
$x_4 = t_2$

$$\Rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -12 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -4 \\ 12 \\ 1 \end{bmatrix}}$$

Problem 6. **(100 points)** Determinants, Chapter 3.
Details 75%, answers 25%.

(a) [20%] Invent a 3×3 non-triangular matrix whose determinant equals $\pi + e^2$. Common approximations are $\pi = 3.14$ and $e = 2.718$, but kindly do not approximate. Expected are determinant evaluation details.

(b) [20%] There are 50 distinct 5×5 matrices A whose entries are restricted to be either 0 or 1. Give one example where $|A| = 0$ and each row and column of A contains at least two zeros and at least two ones. Expected is an explanation for $|A| = 0$.

(c) [60%] Determine all values of x for which A^{-1} exists, where $A = 2I + C$, I is the 3×3 identity and $C = \begin{pmatrix} 1 & x & -1 \\ x & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}$.

a) set top row as (1 0 0) in order to use cofactor expansion to easily calculate determinant:

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 2 & \pi & -e \\ 1 & e & 1 \end{bmatrix}}$$

*other values in column 1 do not affect determinant because $\text{determinant} = +1\begin{vmatrix} \pi & -e \\ e & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -e \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & \pi \\ 1 & e \end{vmatrix}$

$$= +1\begin{vmatrix} \pi & -e \\ e & 1 \end{vmatrix} = \pi(1) - (-e)(e) = \pi + e^2$$

b)
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$|A| =$ (row 1)
$$+1\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$
(row 1 = row 2) (row 1 = row 2)

$= +1(0) - 1(0) = 0$

thm ↑ equal rows ⟹ DET = 0

* ROW 2 & ROW 3 are equivalent, so after completing the rref (A), there will be at least 1 row of zeros. The $|A| = |\text{rref}(A)|$, & if there is a row of zeros the determinant will be 0. (Pick the row of zeros to do cofactor expansion on, and all cross out determinants will be multiplied by 0, so the $|A| = 0$)

→

c) $A = 2I + C = 2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}1&x&-1\\x&0&1\\1&0&-2\end{bmatrix} = \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} + \begin{bmatrix}1&x&-1\\x&0&1\\1&0&-2\end{bmatrix}$

$A = \begin{bmatrix}3&x&-1\\x&2&1\\1&0&0\end{bmatrix}$ * cofactor expansion along 3rd row:

$$|A| = +1\begin{vmatrix}x&-1\\2&1\end{vmatrix} - 0\begin{vmatrix}3&-1\\x&1\end{vmatrix} + 0\begin{vmatrix}3&x\\x&2\end{vmatrix} = \begin{vmatrix}x&-1\\2&1\end{vmatrix} = x+2$$

A^{-1} exists if $|A| \neq 0$

$\Rightarrow |A| \neq 0 \Rightarrow x+2 \neq 0 \Rightarrow \boxed{x \neq -2}$

A^{-1} exists for all x values except $x = -2$