MATH 2270-2 Exam 1 Spring 2018

No books or notes. No electronic devices, please.
Each question has credit 100, with multiple parts given a percentage of the total 100.
If you must write a solution out of order or on the back side, then supply a road map.

Problem 1. (100 points)

Symbol I is used below for the $n \times n$ identity. Notation C^T means the transpose of matrix C. Accept as known theorems the following results:

Theorem 1. If A and B are $n \times n$ and $AB = I$, then $BA = I$.
Theorem 2. If A and B are invertible $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
Theorem 3. If matrices F, G have dimensions allowing FG to be defined, then $(FG)^T = G^T F^T$.
Theorem 4. If matrices F, G have dimensions allowing $F + G$ to be defined, then $(F + G)^T = F^T + G^T$.
Theorem 5. If C is $n \times n$ invertible, then C^T is invertible and $\left(C^T\right)^{-1} = \left(C^{-1}\right)^T$.
Theorem 6. If A and B are $n \times n$, then $|AB| = |A||B|$ (Determinant Product Theorem).
Theorem 7. Assume A is $n \times n$. Matrix A is invertible if and only $|A| \neq 0$.

In each of parts **(a)** and **(b)**, either invent a counter example or else explain why it is true citing the theorems above.

(a) [50%] If matrices A, B are $n \times n$ with $A^T = A$ and A^{-1} exists, then $A(A^{-1} + B)^T = I + (BA)^T$.

(b) [50%] If matrices A, B are $n \times n$ and A^2B^2 is not invertible, then both A and B have determinant zero.

Problem 2. (100 points)

Definition: An **elementary matrix** is the answer after applying exactly one combo, swap or multiply to the identity matrix I. An **elimination matrix** is a product of elementary matrices.

Let A be a 3×6 matrix. Find the elimination matrix E which under left multiplication against matrix A performs (1), (2) and (3) below with one matrix multiply.

(1) Replace Row 3 of A with Row 3 minus Row 1 to obtain new matrix A_1.

(2) Swap Row 2 and Row 3 of A_1 to obtain new matrix A_2.

(3) Multiply Row 2 of A_2 by $1/2$ to obtain new matrix A_3.

Problem 3. (100 points) Let a, b and c denote constants and consider the system of equations

$$\left\{\begin{array}{rcrcrcr} cx & + & by & + & z & = & a \\ (b+c)x & - & ay & + & 2z & = & -a \\ bx & + & ay & + & z & = & -a \end{array}\right.$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a) [40%] The system has a unique solution for $(b-c)(2a+b) \neq 0$.

(b) [30%] The system has no solution if $b + 2a = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c) [30%] The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities).

Definition. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided solving vector equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ results in the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called **dependent**.

Problem 4. (100 points) Solve parts (a), (b) and (c) using the vectors displayed below. Expected is application of the Pivot Theorem or the definition of independence (above). Details are 75%, answer 25%.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}$$

(a) [50%] Show details for the dependence of the 4 vectors.
(b) [20%] List a maximum number of independent vectors extracted from the 4 vectors.
(c) [30%] Explain why the 4 column vectors fail to span the vector space $\mathcal{R}^5$, without using the results from parts (a), (b).

Problem 5. (100 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 4 \\ 0 \end{pmatrix}$$

Expected: (a) [10%] Augmented matrix, (b) [40%] Toolkit steps for the RREF, (c) [10%] Conversion of RREF to scalar equations, (d) [20%] Last frame Algorithm details to write out the scalar general solution, (e) [20%] Conversion of the scalar general solution to the vector general solution. This answer is in the form of a single vector equation for $\vec{x}$, the solution of system $A\vec{x} = \vec{b}$.

Problem 6. (100 points) Determinant problem, chapter 3.

Details 75%, answers 25%.

(a) [30%] Assume given 3×3 matrices A, B. Assume E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap and a multiply by 4. Assume $\det(B) = 5$ and $E_3E_2E_1A = A^2B^3$. Let $C = -2A$. Find all possible values of $\det(C)$.

(b) [30%] Determine all values of x for which A^{-1} exists, where $A = I + C$, I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ x & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(c) [40%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 3, column 2 of A^{-1}.

End Exam 1.