MATH 2270-4 **Exam 1** Spring 2019

NAME: ______________________________

No books or notes. No electronic devices, please.

Each question has credit 100, with multiple parts given a percentage of the total 100.

If you must write a solution out of order or on the back side, then supply a road map.

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QUESTION	VALUE	SCORE
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
TOTAL	600	

Problem 1. (100 points) Matrix Algebra, Chapters 1,2.
Symbol I is used below for the $n \times n$ identity. Notation C^T means the transpose of matrix C. Accept as known theorems the following results:

Theorem 1. If C and D are $n \times n$ and $CD = I$, then $DC = I$.
Theorem 2. If A and B are invertible $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
Theorem 3. If matrices F, G have dimensions allowing FG to be defined, then $(FG)^T = G^T F^T$.
Theorem 4. If C is $n \times n$ invertible, then C^T is invertible and $\left(C^T\right)^{-1} = \left(C^{-1}\right)^T$.

In the statement below, either invent a counter example or else explain why it is true (citing relevant theorems above). Used in the theorems is the definition of inverse: G has an inverse H if and only if $GH = I$ and $HG = I$.

If matrices A, B are $n \times n$ with $A^T A = I$, then A^{-1} exists and $A^{-1}(A + B^T) = I + (BA)^T$.

Problem 2. (100 points) Elementary Matrices and Toolkit Sequences, Chapters 1,2.

> **Definition**: An **elementary matrix** E is the matrix answer after applying exactly one combo, swap or multiply to the identity matrix I. An **elimination matrix** M is a product of elementary matrices.

Let A be a 3×4 matrix. Find the elimination matrix M which under left multiplication against matrix A performs (1), (2) and (3) below with one matrix multiply.

(1) Replace Row 3 of A with Row 3 minus twice Row 2 to obtain new matrix A_1.

(2) Swap Row 1 and Row 3 of A_1 to obtain new matrix A_2.

(3) Multiply Row 3 of A_2 by 1/5 to obtain new matrix A_3.

Problem 3. (100 points) Linear algebraic equations.
System $A\vec{u} = \vec{b}$ with symbols. The Three Possibilities. Chapters 1,2,3.

Let symbols a, b and c denote constants and consider the system of equations

$$\left\{\begin{array}{rcrcrcr} x & + & by & + & cz & = & a \\ 2x & + & (b+c)y & - & az & = & -a \\ x & + & cy & + & az & = & -a \end{array}\right.$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a) [40%] The system has a unique solution for $(c - b)(2a + c) \neq 0$.

(b) [30%] The system has no solution if $2a + c = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c) [30%] The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities).

Definition. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided solving vector equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ results in the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called **dependent**.

Problem 4. (100 points) Linear Independence, Chapters 1,2,3.
Solve parts (a), (b) and (c) using the vectors displayed below. Application of theorems is expected: the Pivot Theorem, the Rank Test, the Determinant Test. Or, directly use the definition of independence (above). Details are 75%, answer 25%.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

(a) [50%] Show details for the dependence of the 4 vectors.
(b) [20%] List a maximum number of independent vectors extracted from the 4 vectors.
(c) [30%] Write each vector not listed in **(b)** as a linear combination of the reported independent vectors.

Problem 5. (100 points) Vector general solution of a matrix equation $A\vec{x} = \vec{b}$, Chapters 1,2.
Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 0 & 1 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 5 \\ 3 \\ 8 \\ 0 \end{pmatrix}$$

Expected: (a) [10%] Augmented matrix, (b) [40%] Toolkit steps for the RREF, (c) [10%] Conversion of RREF to scalar equations, (d) [20%] Last frame Algorithm details to write out the scalar general solution, (e) [20%] Conversion of the scalar general solution to the vector general solution. This answer is in the form of a single vector equation for $\vec{x}$, the solution of system $A\vec{x} = \vec{b}$. The expected components of $\vec{x}$ are x_1, x_2, x_3, x_4.

Problem 6. (100 points) Determinants, Chapter 3.
Details 75%, answers 25%.

(a) [20%] Invent a 3×3 non-triangular matrix whose determinant equals $\pi + e^2$. Common approximations are $\pi = 3.14$ and $e = 2.718$, but kindly do not approximate. Expected are determinant evaluation details.

(b) [20%] There are 50 distinct 5×5 matrices A whose entries are restricted to be either 0 or 1. Give one example where $|A| = 0$ and each row and column of A contains at least two zeros and at least two ones. Expected is an explanation for $|A| = 0$.

(c) [60%] Determine all values of x for which A^{-1} exists, where $A = 2I + C$, I is the 3×3 identity and $C = \begin{pmatrix} 1 & x & -1 \\ x & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}$.

End Exam 1.