Math 310, Lesieutre
Final exam practice
November 25, 2015

Name: ______________________________

This is a version of an old exam, with a couple questions replaced. Note: many topics are still not represented!

Please let me know if you spot any mistakes in my solutions!

1. (a) *Solve the following system of linear equations, putting your answer in parametric vector form:*

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 12 \\ -2x_1 & - & 2x_2 & + & x_3 & & & = & -1 \\ x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 10 \end{array}$$

We set up the augmented matrix and do row reduction:

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ -2 & -2 & 1 & 0 & -1 \\ 1 & 1 & 2 & 1 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 1 & 11 \\ 1 & 1 & 2 & 1 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 1 & 11 \\ 0 & 0 & 1 & 1/2 & 4 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 1 & 11 \\ 0 & 0 & 2 & 1 & 8 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 1 & 11 \\ 0 & 0 & 0 & 1/3 & 2/3 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 1 & 11 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 12 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 2 & 2 & 0 & 10 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

The variable x_2 is free, while the others are pivots. Let s be a parameter. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-s \\ s \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) *Consider the matrix* $A = \begin{bmatrix} 11 & 6 & 17 & 28 \\ -1 & -1 & -2 & -3 \\ 3 & 2 & 5 & 8 \end{bmatrix}$ *with reduced echelon form* $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Given that*

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 186 \\ -21 \\ 54 \end{bmatrix},$$

find the parametric vector form the the solutions to $A\mathbf{x} = \begin{bmatrix} 186 \\ -21 \\ 54 \end{bmatrix}$.

The general solution to $A\mathbf{x} = \mathbf{b}$ is given by the sum of the general solution to $A\mathbf{x} = \mathbf{0}$ and a particular solution, which in this case is the given $\mathbf{x}_p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

The augmented matrix for the corresponding homogeneous system is

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The variables x_3 and x_4 are free, with parameters s and t, and parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-2t \\ -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the general solution to the inhomogeneous system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2. *Suppose that* $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 19 & 2 & 1 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$.

   (a) *What is the determinant of A?*

   Use cofactor expansion on the third row:

   $$\det A = -\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} = -(4+4+2-8-4-1) = -(-3) = 3,$$

   where for the 3×3 determinant I used the special formula.

   (b) *If $A\mathbf{x} = \mathbf{b}$, use Cramer's rule to find x_2.*

   Cramer's rule tells us that

   $$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\det \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 4 & 4 & 2 \\ 0 & 2 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 19 & 2 & 1 \end{bmatrix}} = \frac{6}{3},$$

since the top determinant is just twice the one we just did (as you can see by doing cofactor expansion along the third row again).

3. *Suppose that* $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) *Compute* $B = A^T A$ *and find* B^{-1}.

   We have

$$B = A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

   To get the inverse, we probably just want to row reduce. You could use the adjugate method if you like it, I guess.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -2 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

   The inverse is then

$$B^{-1} = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

   (b) *Explain why* $A\mathbf{x} = \mathbf{b}$ *is inconsistent, and write down the normal equations of the system.*

   Write down the augmented matrix. It's

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

   This is in echelon form, and we have a row of the form $[0\ 0\ 0 \mid b]$ with b not 0. That means the system isn't consistent.

   The normal equations are $A^T A\mathbf{x} = A^T\mathbf{b}$, which is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

(c) *Find the least squares solution to $A\mathbf{x} = \mathbf{b}$.*

From the preceding parts of the question, we get

$$\hat{\mathbf{x}} = B^{-1}\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}6 & -2 & -1\\-2 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}-2\\1\\1\end{bmatrix}$$

(d) *What is the projection of $\mathbf{b}$ onto $\operatorname{Col} A$?*

Well, it's

$$A\hat{\mathbf{x}} = A = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$$

4. (a) *Show that $\mathcal{B} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ are both bases for $\mathbb{R}^2$.*

We really just need to check that the given vectors are linearly independent, since if you have two independent vectors in $\mathbb{R}^2$ they must span and thus be a basis. Solving $B\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{cc|c}1 & 1 & 0\\2 & 1 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 0\\0 & -1 & 0\end{array}\right].$$

This has no free variables, so there are no nonzero solutions, which means tha tthe columns are indpendent as needed. So it's a basis. Similar deal for the other one:

$$C = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

is already in echelon form, so the columns are independent.

(b) *Write down the change of basis matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$.*

We have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\2 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}$$

For the other,

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1} = \begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix}1 & 0\\-2 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}.$$

(c) *If $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}$, find both $[\mathbf{v}]_{\mathcal{C}}$ and $[\mathbf{v}]_{\mathcal{B}}$.*

We have

$$[\mathbf{v}]_{\mathcal{C}} = \mathcal{P}_{\mathcal{C}}^{-1}\mathbf{v} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

For the other one,

$$[\mathbf{v}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}^{-1}\mathbf{v} = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{-1}\begin{bmatrix}1 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}.$$

(d) *Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that $T(\mathbf{c}_1) = \mathbf{b}_1 + 2\mathbf{b}_2$ and $T(\mathbf{c}_2) = \mathbf{b}_2$. Find $[T]_{\mathcal{C}}$, the matrix for T relative to $\mathcal{C}$.*

Our matrix is going to be

$$\big[[T(\mathbf{c}_1)]_{\mathcal{C}}\ [T(\mathbf{c}_2)]_{\mathcal{C}}\big].$$

We know $T(\mathbf{c}_1) = \mathbf{b}_1 + 2\mathbf{b}_2$, which means that

$$[T(\mathbf{c}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using our change of basis,

$$[T(\mathbf{c}_1)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}[T(\mathbf{c}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Similarly,

$$[T(\mathbf{c}_2)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}[T(\mathbf{c}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So our answer is

$$[T]_{\mathcal{C}} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix}.$$

5. (a) *Are the three polynomials $1 + 2t, t + t^2, 3t^2 + 2t - 4$ a basis for $\mathbb{P}_2$? Explain why or why not.*

To check this, you just want to check the corresponding question for the coordinate vectors.

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}.$$

It's a matter of doing row reduction.

$$\begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 10 \\ 0 & 0 & -7 \end{bmatrix}.$$

These are linearly independent, and three linearly independent length-3 vectors are always a basis.

(b) *Every year 20% of the people in City A move to City B, and 10% of the people in City B move to city A. Suppose that initially, each city has $1,000,000$ people. How many people will live in each city after two years? After a very large number of years?*

The stochastic matrix for this process is

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}.$$

We have $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (we'll remember to multiply by a million at the end). After two years

$$\begin{aligned}
\mathbf{x}_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{x}_1 &= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix} \\
\mathbf{x}_1 &= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix} = \begin{bmatrix} 0.83 \\ 1.17 \end{bmatrix}
\end{aligned}$$

In the long run, it's going to approach the steady state. We need a vector in the nullspace of $A - I$, which is

$$\begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}$$

An eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The steady state should have entries adding up to 1, so we want $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$; this corresponds to the proportion of the people who will be in each of the two cities.

How many actual people is that, if there are 2000000 total in circulation? Well, it's in 666666⅔ City A and 1333333⅓ in City B.

6. (a) *Diagonalize the matrix* $A = \begin{bmatrix} 4 & -6 \\ 0 & 1 \end{bmatrix}$.

This matrix is triangular, so we know already what the eigenvalues are: they're $\lambda = 4$ and $\lambda = 1$. For the first, $A - 4I = \begin{bmatrix} 0 & -6 \\ 0 & -3 \end{bmatrix}$, and an eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For the second, $A - I = \begin{bmatrix} 3 & -6 \\ 0 & 0 \end{bmatrix}$, and an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Hence we can diagonalize it as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) *Explain how to quickly compute* A^{20}. *(You don't need to actually do it.)*

It's equal to $PD^{20}P^{-1}$, which is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4^{20} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1}.$$

(c) *Give the solution to the differential equation* $\mathbf{x}' = A\mathbf{x}$ *with* $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The general solution is given by

$$c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}=c_1\begin{bmatrix}1\\0\end{bmatrix}e^{4t}+c_2\begin{bmatrix}2\\1\end{bmatrix}e^{t}$$

Plugging in $t=0$ gives $c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$, and it's easy to see that $c_1=1$, $c_2=1$. So the solution we want is

$$\begin{bmatrix}1\\0\end{bmatrix}e^{4t}+\begin{bmatrix}2\\1\end{bmatrix}e^{t}$$

7. *The matrix*

$$A=\begin{bmatrix}4&20&8&16&16&-3&23\\8&40&5&21&10&-2&36\\5&25&2&12&4&-1&21\\1&5&2&4&4&-1&5\end{bmatrix}$$

can be row reduced to

$$\begin{bmatrix}1&5&0&2&0&0&4\\0&0&1&1&2&0&2\\0&0&0&0&0&1&3\\0&0&0&0&0&0&0\end{bmatrix}.$$

(a) *Give bases for* Row A, Col A, *and* Nul A.

A basis for the row space is the nonzero rows of the echelon form:

$$\begin{bmatrix}1\\5\\0\\2\\0\\0\\4\end{bmatrix},\quad\begin{bmatrix}0\\0\\1\\1\\2\\0\\2\end{bmatrix},\quad\begin{bmatrix}0\\0\\0\\0\\0\\1\\3\end{bmatrix}.$$

A basis for the column space is the columns of the original matrix corresponding to the pivots that we can see in echelon form, namely columns 1, 3, and 6:

$$\begin{bmatrix}4\\8\\5\\1\end{bmatrix},\quad\begin{bmatrix}8\\5\\2\\2\end{bmatrix},\quad\begin{bmatrix}-3\\-2\\-1\\-1\end{bmatrix}.$$

For the nullspace, we have four free variables: x_2, x_4, x_5, and x_7.

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix}=\begin{bmatrix}-5x_2-2x_4-4x_7\\x_2\\-x_4-2x_5-2x_7\\x_4\\x_5\\-3x_7\\x_7\end{bmatrix}=x_2\begin{bmatrix}-5\\1\\0\\0\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-2\\0\\-1\\1\\0\\0\\0\end{bmatrix}+x_5\begin{bmatrix}0\\0\\-2\\0\\1\\0\\0\end{bmatrix}+x_7\begin{bmatrix}-4\\0\\-2\\0\\0\\-3\\1\end{bmatrix}.$$

These four vectors are a basis for the nullspace.

$$\begin{bmatrix}-5\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-2\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\-2\\0\\0\\-3\\1\end{bmatrix}.$$

(b) *What is the rank of A, and what are the dimensions of* $\operatorname{Row} A^T$, $\operatorname{Col} A^T$, *and* $\operatorname{Nul} A^T$*?*

The rank is the dimension of the column space, which is 3. The rank of A^T is equal to the rank of A, again 3. Both the row space and column space of A^T then have dimension 3. We know $\dim \operatorname{Nul} A^T + 3 = \#\text{cols of } A^T = 4$, so $\dim \operatorname{Nul} A^T = 1$.

8. (a) *Find an LU-decomposition of the matrix* $X = \begin{bmatrix}1 & 0 & 2\\2 & -1 & 3\\4 & 1 & 8\end{bmatrix}$.

We need to do row reduction.

$$\begin{bmatrix}1 & 0 & 2\\2 & -1 & 3\\4 & 1 & 8\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & -1 & -1\\4 & 1 & 8\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & -1 & -1\\0 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & -1 & -1\\0 & 0 & -1\end{bmatrix}.$$

The row operations were

- subtract 2 row 1 from row 2;
- subtract 4 row 1 from row 3;
- subtract -1 row 2 from row 3.

Hence the LU decomposition is

$$L = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\4 & -1 & 1\end{bmatrix}, \quad U = \begin{bmatrix}1 & 0 & 2\\0 & -1 & -1\\0 & 0 & -1\end{bmatrix}.$$

(b) *Use your answer to (a) to find the determinant of X.*

We already row-reduced it, so it's just a matter of multiplying together the pivots, which gives $(1)(-1)(-1) = 1$.

9. (a) *Find a QR-decomposition of the matrix* $Y = \begin{bmatrix}1 & 0\\1 & 1\\1 & 2\end{bmatrix}$.

We have to do Gram–Schmidt on the columns of Y.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then we have to normalize the vectors by dividing by the lengths.

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

So our matrix Q is

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}.$$

Then

$$R = Q^T Y = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}.$$

(b) *What is the orthogonal projection of* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *onto* $\operatorname{Col} Y$*?*

We might as well put our orthogonal basis to work. It's

$$\operatorname{proj}_{\operatorname{Col} Y} \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 2/6 \\ 5/6 \end{bmatrix}.$$

10. *Consider the matrix*

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) *What are the eigenvalues of H? Find all of the eigenvectors for each eigenvalue.*

It's lower triangular, so you can see right away what the eigenvalues are: they're 2, 2, and 3. Let's do 3 first.

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Row reduction gives

$$\begin{bmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the nullspace is then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is the eigenvector for eigenvalue 3.

For $\lambda = 2$, we get

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

After a couple row swaps and a division by 3, echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}`.$$

A basis for the nullspace is the single vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (there's only one free variable, so we only get one basis vector).

(b) *Is the matrix H diagonalizable? Explain.*

No, it's not diagonalizable. To diagonalize, we'd need to find two linearly independent eigenvectors for the eigenvalue $\lambda = 2$. But there was only one.