Math 310, Lesieutre Name: ______________________________
Exam 1 Practice
September 23, 2015

This is a version of an old exam replacing some topics we didn't cover yet. Solutions will be posted over the weekend.

Please let me know if you spot any mistakes in my solutions.

1. *Write down all of the solutions (if any) to the following system of linear equations. Give your answer in parametric vector form, and show all of your work.*

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & & + & x_4 & = & 2 \\ 2x_1 & + & 4x_2 & + & x_3 & + & 4x_4 & = & 4 \\ -x_1 & - & 2x_2 & + & 2x_3 & + & 3x_4 & = & -2 \end{array}$$

The augmented matrix for the system is

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 4 & 4 \\ -1 & -2 & 2 & 3 & -2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ -1 & -2 & 2 & 3 & -2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_1 and x_3 are basic variables, while x_2 and x_4 are free. The general solution is

$$\begin{cases} x_1 = 2 - 2x_2 - x_4 \\ x_2 \text{ is free} \\ x_3 = -2x_4 \\ x_4 \text{ is free} \end{cases}$$

Let s be a parameter for the free variable for x_2, and t a parameter for x_4.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2. (a) *The matrix*

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

is invertible. Find A^{-1} by any method you know.

At this point I think we only know one method: using row reduction. We set up the augmented matrix and go until we get the identity on the left.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1/5 & 0 & -2/5 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 & -5 & 15 \\ 0 & 0 & 1 & 0 & 2 & -5 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 2 & -5 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 2 & -5 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 2 & -5 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5 & -14 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 2 & -5 \end{array}\right]$$

The inverse is

$$A^{-1} = \begin{bmatrix} 1 & 5 & -14 \\ 0 & -1 & 3 \\ 0 & 2 & -5 \end{bmatrix}.$$

(b) *Solve the two systems of vector equations* $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ *and* $A\mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

We have

$$\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -14 \\ 0 & -1 & 3 \\ 0 & 2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ -5 \end{bmatrix}$$

and

$$\mathbf{y} = A^{-1}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -14 \\ 0 & -1 & 3 \\ 0 & 2 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix}$$

(c) *Answer True or False, and explain your answer: if a* 7×7 *matrix has linearly independent columns, then the matrix is invertible if and only if the columns span all of* $\mathbb{R}^7$.

It's true, but vacuously: if a matrix is square and has linearly independent columns, then it is automatically invertible, and the columns automatically span all of $\mathbb{R}^7$. This is one of the tests from section 2.3 in the book.

3. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 3 \\ 2 & 2 & 3 & 4 \end{bmatrix}.$$

(a) *Are the columns of A linearly dependent or linearly independent? Justify your answer.*

The columns are linearly dependent: any set of four vectors in $\mathbb{R}^3$ are linearly dependent, so we don't even need to do any calculations to check this.

(b) *For what values of n and m does the matrix A determine a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.*

We can multiply this matrix by a size-4 vector and get back a size-3 vector. That means it determines a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$.

(c) *Is the linear transformation T with standard matrix A one-to-one or onto? Justify your answer.*

To know if it's one-to-one, we need to know if the columns are linearly independent. We already know that they're not. To know if it's onto, we want to know if there's a pivot in every row, which requires us to compute:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 3 & 3 \\ 2 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

To find the pivots, it's enough to just get to echelon form, not rref. There's a pivot in every row, so the map is onto.

4. (a) *Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that sends*

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Write down the standard matrix A for T.

The standard matrix is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Remember that to find it you stick the vectors $T(\mathbf{e}_i)$ in as the columns of the matrix A.

(b) *Compute the following, or explain why the operations don't make sense: AA^T, A^TA, $AA^T + A^TA$.*

$$AA^T = A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 6 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

The first of these is 2×2 while the second is 3×3, so we cant add them.

(c) *What is $\mathbf{b} = T\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$?*

The vector is

$$\mathbf{b} = T\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

(d) *Let* $\mathbf{a} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. *Draw and label the vectors* $\mathbf{a}$, $\mathbf{b}$, $-\mathbf{a}$, *and* $\mathbf{b}-\mathbf{a}$ *on a set of coordinate axes.*

We have

$$\mathbf{a} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$
$$-\mathbf{a} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \qquad \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

5. (a) *Balance the following chemical reaction:*

$$KMnO_4 + MnSO_4 + H_2O \to MnO_2 + K_2SO_4 + H_2SO_4$$

Let's name the coefficients we're looking for:

$$x_1\mathrm{KMnO_4} + x_2\mathrm{MnSO_4} + x_3\mathrm{H_2O} \to x_4\mathrm{MnO_2} + x_5\mathrm{K_2SO_4} + x_6\mathrm{H_2SO_4}$$

Each element gives an equation:

$$\begin{aligned} \mathrm{K}: &\quad x_1 = 2x_5, \\ \mathrm{Mn}: &\quad x_1 + x_2 = x_4, \\ \mathrm{O}: &\quad 4x_1 + 4x_2 + x_3 = 2x_4 + 4x_5 + 4x_6, \\ \mathrm{S}: &\quad x_2 = x_5 + x_6, \\ \mathrm{H}: &\quad 2x_3 = 2x_6. \end{aligned}$$

Then form the augmented matrix. I'm going to use the "O" row at the top, because this way we start a little closer to echelon form.

$$\left[\begin{array}{cccccc|c} 4 & 4 & 1 & -2 & -4 & -4 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -2 & 0 \end{array}\right]$$

After row reduction, you find the solution is:

$$x_1 = 2 \qquad x_2 = 3$$
$$x_3 = 2 \qquad x_4 = 5$$
$$x_5 = 1 \qquad x_6 = 2$$

So the balanced equation is

$$2\mathrm{KMnO_4} + 3\mathrm{MnSO_4} + 2\mathrm{H_2O} \to 5\mathrm{MnO_2} + \mathrm{K_2SO_4} + 2\mathrm{H_2SO_4}$$

(b) *Each year, 50% of the people from City A move to City B, and 30% of the people from City B move to City A. Suppose that both cities have an initial population of 1,000. Calculate the populations of the two cities after 2 years.*

Those are some pretty weird numbers, but let's go with it. We have $\mathbf{x}_{n+1} = A\mathbf{x}_n$, where

$$A = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}.$$

Then

$$\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 800 \\ 1200 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 800 \\ 1200 \end{bmatrix} = \begin{bmatrix} 760 \\ 1240 \end{bmatrix}.$$