10.1 Jordan Form and Eigenanalysis

Generalized Eigenanalysis

The main result is **Jordan's decomposition**

$$A = PJP^{-1},$$

valid for any real or complex square matrix A. We describe here how to compute the invertible matrix P of generalized eigenvectors and the upper triangular matrix J, called a **Jordan form** of A.

Jordan block. An $m \times m$ upper triangular matrix $B(\lambda, m)$ is called a **Jordan block** provided all m diagonal elements are the same eigenvalue λ and all super-diagonal elements are one:

$$B(\lambda, m) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (m \times m \text{ matrix})$$

Jordan form. Given an $n \times n$ matrix A, a **Jordan form** J for A is a block diagonal matrix

$$J = \textbf{diag}(B(\lambda_1, m_1), B(\lambda_2, m_2), \ldots, B(\lambda_k, m_k)),$$

where $\lambda_1, \ldots, \lambda_k$ are eigenvalues of A (duplicates possible) and $m_1 + \cdots + m_k = n$. The eigenvalues of J are on the diagonal of J and J has exactly k eigenpairs. If $k < n$, then J is non-diagonalizable. Relation $AP = PJ$ implies A has exactly k eigenpairs and A fails to be diagonalizable for $k < n$.

The relation $A = PJP^{-1}$ is called a **Jordan decomposition** of A. Invertible matrix P is called the **matrix of generalized eigenvectors** of A. It defines a coordinate system $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ in which the vector function $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is transformed to the simpler vector function $\vec{\mathbf{y}} \to J\vec{\mathbf{y}}$.

If equal eigenvalues are adjacent in J, then Jordan blocks with equal diagonal entries will be adjacent. Zeros can appear on the super-diagonal of J, because adjacent Jordan blocks join on the super-diagonal with a zero. A complete specification of how to build J from A appears below.

Decoding a Jordan Decomposition $A = PJP^{-1}$. If J is a single Jordan block, $J = B(\lambda, m)$, then $P = \langle \vec{\mathbf{v}}_1 | \ldots | \vec{\mathbf{v}}_m \rangle$ and $AP = PJ$

means

$$\begin{array}{rcl} A\vec{\mathbf{v}}_1 & = & \lambda\vec{\mathbf{v}}_1, \\ A\vec{\mathbf{v}}_2 & = & \lambda\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_1, \\ \vdots & \vdots & \vdots \\ A\vec{\mathbf{v}}_m & = & \lambda\vec{\mathbf{v}}_m + \vec{\mathbf{v}}_{m-1}. \end{array}$$

The exploded view of the relation $AP = PB(\lambda, m)$ is called a **Jordan chain**. The formulas can be compacted via matrix $N = A - \lambda I$ into the recursion

$$N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}, \quad N\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1, \ldots, N\vec{\mathbf{v}}_m = \vec{\mathbf{v}}_{m-1}.$$

The first vector $\vec{\mathbf{v}}_1$ is an eigenvector. The remaining vectors $\vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m$ are **not eigenvectors**, they are called **generalized eigenvectors**. A similar formula can be written for each distinct eigenvalue of a matrix A. The collection of formulas are called **Jordan chain relations**. A given eigenvalue may appear multiple times in the chain relations, due to the appearance of two or more Jordan blocks with the same eigenvalue.

Theorem 1 (Jordan Decomposition)
Every $n \times n$ matrix A has a Jordan decomposition $A = PJP^{-1}$.

Proof: The result holds by default for 1×1 matrices. Assume the result holds for all $k \times k$ matrices, $k < n$. The proof proceeds by induction on n.

The induction assumes that for any $k \times k$ matrix A, there is a Jordan decomposition $A = PJP^{-1}$. Then the columns of P satisfy Jordan chain relations

$$A\vec{\mathbf{x}}_i^j = \lambda_i \vec{\mathbf{x}}_i^j + \vec{\mathbf{x}}_i^{j-1}, \quad j > 1, \quad A\vec{\mathbf{x}}_i^1 = \lambda_i \vec{\mathbf{x}}_i^1.$$

Conversely, if the Jordan chain relations are satisfied for k independent vectors $\{\vec{\mathbf{x}}_i^j\}$, then the vectors form the columns of an invertible matrix P such that $A = PJP^{-1}$ with J in Jordan form. The induction step centers upon producing the chain relations and proving that the n vectors are independent.

Let B be $n \times n$ and λ_0 an eigenvalue of B. The Jordan chain relations hold for $A = B$ if and only if they hold for $A = B - \lambda_0 I$. Without loss of generality, we can assume 0 is an eigenvalue of B.

Because B has 0 as an eigenvalue, then $p = \dim(\mathbf{kernel}(B)) > 0$ and $k = \dim(\mathbf{Image}(B)) < n$, with $p + k = n$. If $k = 0$, then $B = 0$, which is a Jordan form, and there is nothing to prove. Assume henceforth p and k positive. Let $S = \langle\, \mathbf{col}(B, i_1) | \ldots | \mathbf{col}(B, i_k) \rangle$ denote the matrix of pivot columns $i_1, \ldots, i_k$ of B. The pivot columns are known to span $\mathbf{Image}(B)$. Let A be the $k \times k$ basis representation matrix defined by the equation $BS = SA$, or equivalently, $B\,\mathbf{col}(S, j) = \sum_{i=1}^k a_{ij}\,\mathbf{col}(S, i)$. The induction hypothesis applied to A implies there is a basis of k-vectors satisfying Jordan chain relations

$$A\vec{\mathbf{x}}_i^j = \lambda_i \vec{\mathbf{x}}_i^j + \vec{\mathbf{x}}_i^{j-1}, \quad j > 1, \quad A\vec{\mathbf{x}}_i^1 = \lambda_i \vec{\mathbf{x}}_i^1.$$

The values λ_i, $i = 1, \ldots, p$, are the distinct eigenvalues of A. Apply S to these equations to obtain for the n-vectors $\vec{\mathbf{y}}_i^j = S\vec{\mathbf{x}}_i^j$ the Jordan chain relations

$$B\vec{\mathbf{y}}_i^j = \lambda_i \vec{\mathbf{y}}_i^j + \vec{\mathbf{y}}_i^{j-1}, \quad j > 1, \quad B\vec{\mathbf{y}}_i^1 = \lambda_i \vec{\mathbf{y}}_i^1.$$

Because S has independent columns and the k-vectors $\vec{\mathbf{x}}_i^j$ are independent, then the n-vectors $\vec{\mathbf{y}}_i^j$ are independent.

The **plan** is to isolate the chains for eigenvalue zero, then extend these chains by one vector. Then 1-chains will be constructed from eigenpairs for eigenvalue zero to make n generalized eigenvectors.

Suppose q values of i satisfy $\lambda_i = 0$. We allow $q = 0$. For simplicity, assume such values i are $i = 1, \ldots, q$. The key formula $\vec{\mathbf{y}}_i^j = S\vec{\mathbf{x}}_i^j$ implies $\vec{\mathbf{y}}_i^j$ is in $\mathbf{Image}(B)$, while $B\vec{\mathbf{y}}_i^1 = \lambda_i \vec{\mathbf{y}}_i^1$ implies $\vec{\mathbf{y}}_1^1, \ldots, \vec{\mathbf{y}}_q^1$ are in $\mathbf{kernel}(B)$. Each eigenvector $\vec{\mathbf{y}}_i^1$ starts a Jordan chain ending in $\vec{\mathbf{y}}_i^{m(i)}$. Then[1] the equation $B\vec{\mathbf{u}} = \vec{\mathbf{y}}_i^{m(i)}$ has an n-vector solution $\vec{\mathbf{u}}$. We label $\vec{\mathbf{u}} = \vec{\mathbf{y}}_i^{m(i)+1}$. Because $\lambda_i = 0$, then $B\vec{\mathbf{u}} = \lambda_i \vec{\mathbf{u}} + \vec{\mathbf{y}}_i^{m(i)}$ results in an extended Jordan chain

$$\begin{array}{lcll}
B\vec{\mathbf{y}}_i^1 & = & \lambda_i \vec{\mathbf{y}}_i^1 & \\
B\vec{\mathbf{y}}_i^2 & = & \lambda_i \vec{\mathbf{y}}_i^2 & + \quad \vec{\mathbf{y}}_i^1 \\
 & \vdots & & \\
B\vec{\mathbf{y}}_i^{m(i)} & = & \lambda_i \vec{\mathbf{y}}_i^{m(i)} & + \quad \vec{\mathbf{y}}_i^{m(i)-1} \\
B\vec{\mathbf{y}}_i^{m(i)+1} & = & \lambda_i \vec{\mathbf{y}}_i^{m(i)+1} & + \quad \vec{\mathbf{y}}_i^{m(i)}
\end{array}$$

Let's extend the independent set $\{\vec{\mathbf{y}}_i^1\}_{i=1}^q$ to a basis of $\mathbf{kernel}(B)$ by adding $s = n - k - q$ additional independent vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_s$. This basis consists of eigenvectors of B for eigenvalue 0. Then the set of n vectors $\vec{\mathbf{v}}_r$, $\vec{\mathbf{y}}_i^j$ for $1 \le r \le s$, $1 \le i \le p$, $1 \le j \le m(i) + 1$ consists of eigenvectors of B and vectors that satisfy Jordan chain relations. These vectors are columns of a matrix $\mathcal{P}$ that satisfies $B\mathcal{P} = \mathcal{P}\mathcal{J}$ where $\mathcal{J}$ is a Jordan form.

To prove $\mathcal{P}$ invertible, assume a linear combination of the columns of $\mathcal{P}$ is zero:

$$\sum_{i=q+1}^{p} \sum_{j=1}^{m(i)} b_i^j \vec{\mathbf{y}}_i^j + \sum_{i=1}^{q} \sum_{j=1}^{m(i)+1} b_i^j \vec{\mathbf{y}}_i^j + \sum_{i=1}^{s} c_i \vec{\mathbf{v}}_i = \vec{\mathbf{0}}.$$

Apply B to this equation. Because $B\vec{\mathbf{w}} = \vec{\mathbf{0}}$ for any $\vec{\mathbf{w}}$ in $\mathbf{kernel}(B)$, then

$$\sum_{i=q+1}^{p} \sum_{j=1}^{m(i)} b_i^j B\vec{\mathbf{y}}_i^j + \sum_{i=1}^{q} \sum_{j=2}^{m(i)+1} b_i^j B\vec{\mathbf{y}}_i^j = \vec{\mathbf{0}}.$$

The Jordan chain relations imply that the k vectors $B\vec{\mathbf{y}}_i^j$ in the linear combination consist of $\lambda_i \vec{\mathbf{y}}_i^j + \vec{\mathbf{y}}_i^{j-1}$, $\lambda_i \vec{\mathbf{y}}_i^1$, $i = q+1, \ldots, p$, $j = 2, \ldots, m(i)$, plus the vectors $\vec{\mathbf{y}}_i^j$, $1 \le i \le q$, $1 \le j \le m(i)$. Independence of the original k vectors $\{\vec{\mathbf{y}}_i^j\}$ plus $\lambda_i \ne 0$ for $i > q$ implies this new set is independent. Then all coefficients in the linear combination are zero.

The first linear combination then reduces to $\sum_{i=1}^q b_i^1 \vec{\mathbf{y}}_i^1 + \sum_{i=1}^s c_i \vec{\mathbf{v}}_i = \vec{\mathbf{0}}$. Independence of the constructed basis for $\mathbf{kernel}(B)$ implies $b_i^1 = 0$ for $1 \le i \le q$ and $c_i = 0$ for $1 \le i \le s$. Therefore, the columns of $\mathcal{P}$ are independent. The induction is complete.

[1] The n-vector $\vec{\mathbf{u}}$ is constructed by setting $\vec{\mathbf{u}} = \vec{\mathbf{0}}$, then copy components of k-vector $\vec{\mathbf{x}}_i^{m(i)}$ into pivot locations: $\mathbf{row}(\vec{\mathbf{u}}, i_j) = \mathbf{row}(\vec{\mathbf{x}}_i^{m(i)}, j)$, $j = 1, \ldots, k$.

Geometric and algebraic multiplicity. The **geometric multiplicity** is defined by $\textsf{\textbf{GeoMult}}(\lambda) = \dim(\textbf{kernel}(A - \lambda I))$, which is the number of basis vectors in a solution to $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$, or, equivalently, the number of free variables. The **algebraic multiplicity** is the integer $k = \textsf{\textbf{AlgMult}}(\lambda)$ such that $(r - \lambda)^k$ divides the characteristic polynomial $\det(A - \lambda I)$, but larger powers do not.

Theorem 2 (Algebraic and Geometric Multiplicity)
Let A be a square real or complex matrix. Then

$$1 \le \textsf{\textbf{GeoMult}}(\lambda) \le \textsf{\textbf{AlgMult}}(\lambda). \tag{1}$$

In addition, there are the following relationships between the Jordan form J and algebraic and geometric multiplicities.

$\textsf{\textbf{GeoMult}}(\lambda)$	Equals the number of Jordan blocks in J with eigenvalue λ,
$\textsf{\textbf{AlgMult}}(\lambda)$	Equals the number of times λ is repeated along the diagonal of J.

Proof: Let $d = \textsf{\textbf{GeoMult}}(\lambda_0)$. Construct a basis $v_1, \ldots, v_n$ of $\mathcal{R}^n$ such that $v_1, \ldots, v_d$ is a basis for $\textbf{kernel}(A - \lambda_0 I)$. Define $S = \langle v_1 | \ldots | v_n \rangle$ and $B = S^{-1}AS$. The first d columns of AS are $\lambda_0 v_1, \ldots, \lambda_0 v_d$. Then $B = \left(\begin{array}{c|c} \lambda_0 I & C \\ \hline 0 & D \end{array}\right)$ for some matrices C and D. Cofactor expansion implies some polynomial g satisfies

$$\det(A - \lambda I) = \det(S(B - \lambda I)S^{-1}) = \det(B - \lambda I) = (\lambda - \lambda_0)^d g(\lambda)$$

and therefore $d \le \textsf{\textbf{AlgMult}}(\lambda_0)$. Other details of proof are left to the reader.

Chains of generalized eigenvectors. Given an eigenvalue λ of the matrix A, the topic of generalized eigenanalysis determines a Jordan block $B(\lambda, m)$ in J by finding an m-**chain** of generalized eigenvectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_m$, which appear as columns of P in the relation $A = PJP^{-1}$. The very first vector $\vec{\mathbf{v}}_1$ of the chain is an eigenvector, $(A - \lambda I)\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. The others $\vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$ are not eigenvectors but satisfy

$$(A - \lambda I)\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1, \quad \ldots \quad , \quad (A - \lambda I)\vec{\mathbf{v}}_m = \vec{\mathbf{v}}_{m-1}.$$

Implied by the term m-**chain** is insolvability of $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{v}}_m$. The chain size m is subject to the inequality $1 \le m \le \textsf{\textbf{AlgMult}}(\lambda)$.

The Jordan form J may contain several Jordan blocks for one eigenvalue λ. To illustrate, if J has only one eigenvalue λ and $\textsf{\textbf{AlgMult}}(\lambda) = 3$, then J might be constructed as follows:

$$\begin{aligned}
J = \textbf{diag}(B(\lambda,1),B(\lambda,1),B(\lambda,1)) &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\
J = \textbf{diag}(B(\lambda,1),B(\lambda,2)) &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \\
J = B(\lambda,3) &= \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.
\end{aligned}$$

The three generalized eigenvectors for this example correspond to

$$\begin{aligned}
J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} &\quad\leftrightarrow\quad \text{Three 1-chains,} \\
J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} &\quad\leftrightarrow\quad \text{One 1-chain and one 2-chain,} \\
J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} &\quad\leftrightarrow\quad \text{One 3-chain.}
\end{aligned}$$

Computing m-chains. Let us fix the discussion to an eigenvalue λ of A. Define $N = A - \lambda I$ and $p = \textbf{AlgMult}(\lambda)$.

To compute an m-chain, start with an eigenvector $\vec{\mathbf{v}}_1$ and solve recursively by `rref` methods $N\vec{\mathbf{v}}_{j+1} = \vec{\mathbf{v}}_j$ until there fails to be a solution. This must seemingly be done for *all possible choices* of $\vec{\mathbf{v}}_1$! The search for m-chains terminates when p independent generalized eigenvectors have been calculated.

If A has an essentially unique eigenpair $(\lambda, \vec{\mathbf{v}}_1)$, then this process terminates immediately with an m-chain where $m = p$. The chain produces one Jordan block $B(\lambda, m)$ and the generalized eigenvectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_m$ are recorded into the matrix P.

If $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$ form a basis for the eigenvectors of A corresponding to λ, then the problem $N\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has 2 free variables. Therefore, we seek to find an m_1-chain and an m_2-chain such that $m_1 + m_2 = p$, corresponding to two Jordan blocks $B(\lambda, m_1)$ and $B(\lambda, m_2)$.

To understand the logic applied here, the reader should verify that for $\mathcal{N} = \textbf{diag}(B(0,m_1), B(0,m_2), \ldots, B(0,m_k))$ the problem $\mathcal{N}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has k free variables, because $\mathcal{N}$ is already in `rref` form. These remarks imply that a k-dimensional basis of eigenvectors of A for eigenvalue λ

causes a search for m_i-chains, $1 \le i \le k$, such that $m_1 + \cdots + m_k = p$, corresponding to k Jordan blocks $B(\lambda, m_1)$, …, $B(\lambda, m_k)$.

A common naive approach for computing generalized eigenvectors can be illustrated by letting

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vec{\mathbf{u}}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Matrix A has one eigenvalue $\lambda = 1$ and two eigenpairs $(1, \vec{\mathbf{u}}_1)$, $(1, \vec{\mathbf{u}}_2)$. Starting a chain calculation with $\vec{\mathbf{v}}_1$ equal to either $\vec{\mathbf{u}}_1$ or $\vec{\mathbf{u}}_2$ gives a 1-chain. This naive approach leads to only two independent generalized eigenvectors. However, the calculation must proceed until three independent generalized eigenvectors have been computed. To resolve the trouble, keep a 1-chain, say the one generated by $\vec{\mathbf{u}}_1$, and start a new chain calculation using $\vec{\mathbf{v}}_1 = a_1\vec{\mathbf{u}}_1 + a_2\vec{\mathbf{u}}_2$. Adjust the values of a_1, a_2 until a 2-chain has been computed:

$$\langle A - \lambda I | \vec{\mathbf{v}}_1 \rangle = \begin{pmatrix} 0 & 1 & 1 & a_1 \\ 0 & 0 & 0 & -a_1 + a_2 \\ 0 & 0 & 0 & a_1 - a_2 \end{pmatrix} \approx \begin{pmatrix} 0 & 1 & 1 & a_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

provided $a_1 - a_2 = 0$. Choose $a_1 = a_2 = 1$ to make $\vec{\mathbf{v}}_1 = \vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2 \ne \vec{\mathbf{0}}$ and solve for $\vec{\mathbf{v}}_2 = \begin{pmatrix} 0, 1, 0 \end{pmatrix}$. Then $\vec{\mathbf{u}}_1$ is a 1-chain and $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ is a 2-chain. The generalized eigenvectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent and form the columns of P while $J = \textbf{diag}(B(\lambda, 1), B(\lambda, 2))$ (recall $\lambda = 1$). We justify $A = PJP^{-1}$ by testing $AP = PJ$, using the formulas

$$J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Jordan Decomposition using `maple`

Displayed here is `maple` code which applied to the matrix

$$A = \begin{pmatrix} 4 & -2 & 5 \\ -2 & 4 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

produces the Jordan decomposition

$$A = PJP^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 4 & -7 \\ -1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 8 & -8 & 16 \\ 2 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

```
A := Matrix([[4, -2, 5], [-2, 4, -3], [0, 0, 2]]);
factor(LinearAlgebra[CharacteristicPolynomial](A,lambda));
# Answer == (lambda-6)*(lambda-2)^2
J,P:=LinearAlgebra[JordanForm](A,output=['J','Q']);
zero:=A.P-P.J; # zero matrix expected
```

Number of Jordan Blocks

In calculating generalized eigenvectors of A for eigenvalue λ, it is possible to decide in advance how many Jordan chains of size k should be computed. A practical consequence is to organize the computation for certain chain sizes.

Theorem 3 (Number of Jordan Blocks)
Given eigenvalue λ of A, define $N = A - \lambda I$, $k(j) = \dim(\mathbf{kernel}(N^j))$. Let p be the least integer such that $N^p = N^{p+1}$. Then the Jordan form of A has $2k(j-1) - k(j-2) - k(j)$ Jordan blocks $B(\lambda, j-1)$, $j = 3, \ldots, p$.

The proof of the theorem is in the exercises, where more detail appears for $p = 1$ and $p = 2$. Complete results are in the `maple` code below.

An Illustration. This example is a 5×5 matrix A with one eigenvalue $\lambda = 2$ of multiplicity 5. Let $s(j)$ = number of $j \times j$ Jordan blocks.

$$A = \begin{pmatrix} 3 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ -3 & 3 & 0 & -2 & 3 \end{pmatrix}, \quad N = A - 2I = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ -3 & 3 & 0 & -2 & 1 \end{pmatrix}.$$

Then $N^3 = N^4 = N^5 = 0$ implies $k(3) = k(4) = k(5) = 5$. Further, $k(2) = 4$, $k(1) = 2$. Then $s(5) = s(4) = 0$, $s(3) = s(2) = 1$, $s(1) = 0$, which implies one block of each size 2 and 3.

Some `maple` code automates the investigation:

```
with(LinearAlgebra):
A := Matrix([
[ 3, -1, 1,  0, 0],[ 2,  0, 1,  1, 0],
[ 1, -1, 2,  1, 0],[-1,  1, 0,  2, 1],
[-3,  3, 0, -2, 3] ]);
lambda:=2;
n:=RowDimension(A);N:=A-lambda*IdentityMatrix(n);
for j from 1 to n do
 k[j]:=n-Rank(N^j); od:
for p from n to 2 by -1 do
```

```
 if(k[p]<>k[p-1])then break; fi: od;
txt:=(j,x)->printf(`if`(x=1,
 cat("B(lambda,",j,") occurs 1 time\n"),
 cat("B(lambda,",j,") occurs ",x," times\n"))):
printf("lambda=%d, nilpotency=%d\n",lambda,p);
if(p=1) then txt(1,k[1]); else
 txt(p,k[p]-k[p-1]);
 for j from p to 3 by -1 do
   txt(j-1,2*k[j-1]-k[j-2]-k[j]): od:
 txt(1,2*k[1]-k[2]);
fi:
#lambda=2, nilpotency=3
#B(lambda,3) occurs 1 time
#B(lambda,2) occurs 1 time
#B(lambda,1) occurs 0 times
J,P:=JordanForm(A,output=['J','Q'])}:
# Answer check for the maple code
```

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 0 & 1 & 2 & -1 & 0 \\ -4 & 2 & 2 & -2 & 2 \\ -4 & 1 & 1 & -1 & 1 \\ -4 & -3 & 1 & -1 & 1 \\ 4 & -5 & -3 & 1 & -3 \end{pmatrix}$$

Numerical Instability

The matrix $A = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}$ has two possible Jordan forms

$$J(\varepsilon) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \varepsilon = 0, \\ \\ \begin{pmatrix} 1+\sqrt{\varepsilon} & 0 \\ 0 & 1-\sqrt{\varepsilon} \end{pmatrix} & \varepsilon > 0. \end{cases}$$

When $\varepsilon \approx 0$, then numerical algorithms become unstable, unable to lock onto the correct Jordan form. Briefly, $\lim_{\varepsilon \to 0} J(\varepsilon) \neq J(0)$.

The Real Jordan Form of A

Given a real matrix A, generalized eigenanalysis seeks to find a *real* invertible matrix $\mathcal{P}$ and a *real* upper triangular block matrix R such that $A = \mathcal{P}R\mathcal{P}^{-1}$.

If λ is a real eigenvalue of A, then a **real Jordan block** is a matrix

$$B = \textbf{diag}(\lambda, \ldots, \lambda) + N, \quad N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If $\lambda = a + ib$ is a complex eigenvalue of A, then symbols λ, 1 and 0 are replaced respectively by 2×2 real matrices $\Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\mathcal{I} = \textbf{diag}(1,1)$ and $\mathcal{O} = \textbf{diag}(0,0)$. The corresponding $2m\times 2m$ real Jordan block matrix is given by the formula

$$B = \textbf{diag}(\Lambda, \ldots, \Lambda) + \mathcal{N}, \quad \mathcal{N} = \begin{pmatrix} \mathcal{O} & \mathcal{I} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{pmatrix}.$$

Direct Sum Decomposition

The **generalized eigenspace** of eigenvalue λ of an $n\times n$ matrix A is the subspace $\mathbf{kernel}((A-\lambda I)^p)$ where $p = \textbf{AlgMult}(\lambda)$. We state without proof the main result for generalized eigenspace bases, because details appear in the exercises. A proof is included for the direct sum decomposition, even though Putzer's spectral theory independently produces the same decomposition.

Theorem 4 (Generalized Eigenspace Basis)
The subspace $\mathbf{kernel}((A-\lambda I)^k)$, $k = \textbf{AlgMult}(\lambda)$ has a k-dimensional basis whose vectors are the columns of P corresponding to blocks $B(\lambda, j)$ of J, in Jordan decomposition $A = PJP^{-1}$.

Theorem 5 (Direct Sum Decomposition)
Given $n\times n$ matrix A and distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $n_1 = \textbf{AlgMult}(\lambda_i)$, $\ldots$, $n_k = \textbf{AlgMult}(\lambda_i)$, then A induces a direct sum decomposition

$$\mathcal{C}^n = \mathbf{kernel}((A-\lambda_1 I)^{n_1} \oplus \cdots \oplus \mathbf{kernel}((A-\lambda_k I)^{n_k}.$$

This equation means that each complex vector $\vec{\mathbf{x}}$ in $\mathcal{C}^n$ can be uniquely written as

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \cdots + \vec{\mathbf{x}}_k$$

where each $\vec{\mathbf{x}}_i$ belongs to $\mathbf{kernel}\left((A-\lambda_i)^{n_i}\right)$, $i = 1, \ldots, k$.

Proof: The previous theorem implies there is a basis of dimension n_i for $E_i \equiv \mathbf{kernel}((A-\lambda_i I)^{n_i})$, $i = 1, \ldots, k$. Because $n_1 + \cdots + n_k = n$, then there are n vectors in the union of these bases. The independence test for these n vectors amounts to showing that $\vec{\mathbf{x}}_1 + \cdots + \vec{\mathbf{x}}_k = \vec{\mathbf{0}}$ with $\vec{\mathbf{x}}_i$ in E_i, $i = 1, \ldots, k$, implies all $\vec{\mathbf{x}}_i = \vec{\mathbf{0}}$. This will be true provided $E_i \cap E_j = \{\vec{\mathbf{0}}\}$ for $i \neq j$.

Let's assume a Jordan decomposition $A = PJP^{-1}$. If $\vec{\mathbf{x}}$ is common to both E_i and E_j, then basis expansion of $\vec{\mathbf{x}}$ in both subspaces implies a linear combination of the columns of P is zero, which by independence of the columns of P implies $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

The proof is complete.

Computing Exponential Matrices

Discussed here are methods for finding a real exponential matrix e^{At} when A is real. Two formulas are given, one for a real eigenvalue and one for a complex eigenvalue. These formulas supplement the spectral formulas given earlier in the text.

Nilpotent matrices. A matrix N which satisfies $N^p = 0$ for some integer p is called **nilpotent**. The least integer p for which $N^p = 0$ is called the **nilpotency** of N. A nilpotent matrix N has a finite exponential series:

$$e^{Nt} = I + Nt + N^2\frac{t^2}{2!} + \cdots + N^{p-1}\frac{t^{p-1}}{(p-1)!}.$$

If $N = B(\lambda, p) - \lambda I$, then the finite sum has a splendidly simple expression. Due to $e^{\lambda t + Nt} = e^{\lambda t}e^{Nt}$, this proves the following result.

Theorem 6 (Exponential of a Jordan Block Matrix)
If λ is real and

$$B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (p \times p \text{ matrix})$$

then

$$e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{p-2}}{(p-2)!} & \frac{t^{p-1}}{(p-1)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The equality also holds if λ is a complex number, in which case both sides of the equation are complex.

Real Exponentials for Complex λ. A Jordan decomposition $A = \mathcal{P}J\mathcal{P}^{-1}$, in which A has only real eigenvalues, has real generalized eigenvectors appearing as columns in the matrix $\mathcal{P}$, in the natural order given in J. When $\lambda = a + ib$ is complex, $b > 0$, then the real and imaginary parts of each generalized eigenvector are entered pairwise into $\mathcal{P}$; the conjugate eigenvalue $\overline{\lambda} = a - ib$ is skipped. The complex entry along the diagonal of J is changed into a 2×2 matrix under the correspondence

$$a + ib \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The result is a *real* matrix $\mathcal{P}$ and a *real* block upper triangular matrix J which satisfy $A = \mathcal{P}J\mathcal{P}^{-1}$.

Theorem 7 (Real Block Diagonal Matrix, Eigenvalue $a + ib$)

Let $\Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\mathcal{I} = \mathbf{diag}(1,1)$ and $\mathcal{O} = \mathbf{diag}(0,0)$. Consider a real Jordan block matrix of dimension $2m \times 2m$ given by the formula

$$B = \begin{pmatrix} \Lambda & \mathcal{I} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \Lambda & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \Lambda \end{pmatrix}.$$

If $\mathcal{R} = \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$, then

$$e^{Bt} = e^{at} \begin{pmatrix} \mathcal{R} & t\mathcal{R} & \frac{t^2}{2}\mathcal{R} & \cdots & \frac{t^{m-2}}{(m-2)!}\mathcal{R} & \frac{t^{m-1}}{(m-1)!}\mathcal{R} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{R} & t\mathcal{R} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{R} \end{pmatrix}.$$

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. The solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ must be real if A is real. The real solution can be expressed as $\vec{\mathbf{x}}(t) = \mathcal{P}\vec{\mathbf{y}}(t)$ where $\vec{\mathbf{y}}'(t) = R\vec{\mathbf{y}}(t)$ and R is a real Jordan form of A, containing real Jordan blocks $B_1, \ldots, B_k$ down its diagonal. Theorems above provide explicit formulas for the block matrices $e^{B_i t}$ in the relation

$$e^{Rt} = \mathbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right).$$

The resulting formula

$$\vec{\mathbf{x}}(t) = \mathcal{P}e^{Rt}\mathcal{P}^{-1}\vec{\mathbf{x}}(0)$$

contains only real numbers, real exponentials, plus sine and cosine terms, which are possibly multiplied by polynomials in t.

Exercises 10.1

Jordan block. Write out explicitly.

1.

2.

3.

4.

Jordan form. Which are Jordan forms and which are not? Explain.

5.

6.

7.

8.

Decoding $A = PJP^{-1}$. Decode $A = PJP^{-1}$ in each case, displaying explicitly the Jordan chain relations.

9.

10.

11.

12.

Geometric multiplicity. Determine the geometric multiplicity **GeoMult**(λ).

13.

14.

15.

16.

Algebraic multiplicity. Determine the algebraic multiplicity **AlgMult**(λ).

17.

18.

19.

20.

Generalized eigenvectors. Find all generalized eigenvectors and represent $A = PJP^{-1}$.

21.

22.

23.

24.

25.

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32.

Computing m-chains. Find the Jordan chains for the given eigenvalue.

33.

34.

35.

36.

37.

38.

39.

40.

Jordan Decomposition. Use `maple` to find the Jordan decomposition.

41.

42.

43.

44.

45.

46.

47.

48.

Number of Jordan Blocks. Outlined here is the derivation of

$$s(j) = 2k(j-1) - k(j-2) - k(j).$$

Definitions:

- $s(j)$= number of blocks $B(\lambda, j)$
- $N = A - \lambda I$
- $k(j) = \dim(\mathbf{kernel}(N^j))$
- $L_j = \mathbf{kernel}(N^{j-1})^{\perp}$ relative to $\mathbf{kernel}(N^j)$
- $\ell(j) = \dim(L_j)$
- p minimizes $\mathbf{kernel}(N^p) = \mathbf{kernel}(N^{p+1})$

49. Verify $k(j) \le k(j+1)$ from

$$\mathbf{kernel}(N^j) \subset \mathbf{kernel}(N^{j+1}).$$

50. Verify the direct sum formula

$$\mathbf{kernel}(N^j) = \mathbf{kernel}(N^{j-1}) \oplus L_j.$$

Then $k(j) = k(j-1) + \ell(j)$.

51. Given $N^j\vec{\mathbf{v}} = \vec{\mathbf{0}}$, $N^{j-1}\vec{\mathbf{v}} \ne \vec{\mathbf{0}}$, define $\vec{\mathbf{v}}_i = N^{j-i}\vec{\mathbf{v}}$, $i = 1, \ldots, j$. Show that these are independent vectors satisfying Jordan chain relations $N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $N\vec{\mathbf{v}}_{i+i} = \vec{\mathbf{v}}_i$.

52. A block $B(\lambda, p)$ corresponds to a Jordan chain $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ constructed from the Jordan decomposition. Use $N^{j-1}\vec{\mathbf{v}}_j = \vec{\mathbf{v}}_1$ and $\mathbf{kernel}(N^p) = \mathbf{kernel}(N^{p+1})$ to show that the number of such blocks $B(\lambda, p)$ is $\ell(p)$. Then for $p > 1$, $s(p) = k(p) - k(p-1)$.

53. Show that $\ell(j-1) - \ell(j)$ is the number of blocks $B(\lambda, j)$ for $2 < j < p$. Then

$$s(j) = 2k(j-1) - k(j) - k(j-2).$$

54. Test the formulas above on the special matrices

$$A = \mathbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,1)),$$
$$A = \mathbf{diag}(B(\lambda,1), B(\lambda,2), B(\lambda,3)),$$
$$A = \mathbf{diag}(B(\lambda,1), B(\lambda,3), B(\lambda,3)),$$
$$A = \mathbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,3)),$$

Generalized Eigenspace Basis.

Let A be $n \times n$ with distinct eigenvalues λ_i, $n_i = \mathbf{AlgMult}(\lambda_i)$ and $E_i = \mathbf{kernel}((A - \lambda_i I)^{n_i})$, $i = 1, \ldots, k$. Assume a Jordan decomposition $A = PJP^{-1}$.

55. Let Jordan block $B(\lambda, j)$ appear in J. Prove that a Jordan chain corresponding to this block is a set of j independent columns of P.

56. Let $\mathcal{B}_\lambda$ be the union of all columns of P originating from Jordan chains associated with Jordan blocks $B(\lambda, j)$. Prove that $\mathcal{B}_\lambda$ is an independent set.

57. Verify that $\mathcal{B}_\lambda$ has $\mathbf{AlgMult}(\lambda)$ basis elements.

58. Prove that $E_i = \mathbf{span}(\mathcal{B}_{\lambda_i})$ and $\dim(E_i) = n_i$, $i = 1, \ldots, k$.

Numerical Instability. Show directly that $\lim_{\epsilon \to 0} J(\epsilon) \ne J(0)$.

59.

60.

61.

62.

Direct Sum Decomposition. Display the direct sum decomposition.

63.

64.

65.

66.

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69.

70.

Exponential Matrices. Compute the exponential matrix on paper and then check the answer using `maple`.

71.

72.

73.

74.

75.

76.

77.

78.

Nilpotent matrices. Find the nilpotency of N.

79.

80.

81.

82.

Real Exponentials. Compute the real exponential e^{At} on paper. Check the answer in `maple`.

83.

84.

85.

86.

Real Jordan Form. Find the real Jordan form.

87.

88.

89.

90.

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Solve the differential equation.

91.

92.

93.

94.