# Fundamental Theorem of Linear Algebra

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## Orthogonality

**Definition 1 (Orthogonal Vectors)**
Two vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ are said to be **orthogonal** provided their dot product is zero:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0.$$

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If both vectors are nonzero (not required in the definition), then the angle $\theta$ between the two vectors is determined by

$$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|} = 0,$$

which implies $\theta = 90^\circ$. In short, orthogonal vectors form a right angle.

## Orthogonal and Orthonormal Set

**Definition 2 (Orthogonal Set of Vectors)**
A given set of nonzero vectors $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_k$ that satisfies the **orthogonality condition**

$$\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_j = 0, \quad i \neq j,$$

is called an **orthogonal set**.

**Definition 3 (Orthonormal Set of Vectors)**
A given set of **unit vectors** $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_k$ that satisfies the **orthogonality condition** is called an **orthonormal set**.

## Orthogonal Complement $W^\perp$ of a Subspace $W$

**Definition**. Let $W$ be a subspace of an inner product space $V$, inner product $\langle \vec{u}, \vec{v} \rangle$. The **orthogonal complement** of $W$, denoted $W^\perp$, is the set of all vectors $\vec{v}$ in $V$ such that $\langle \vec{u}, \vec{v} \rangle = 0$ for all $\vec{u}$ in $W$. In set notation:

$$W^\perp = \{\vec{v} \,:\, \langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \text{ in } W\}$$

**Example**. If $V = R^3$ and $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$, then $W^\perp$ is the span of the calculus/physics cross product $\vec{u}_1 \times \vec{u}_2$. The equation $\dim(W) + \dim(W^\perp) = 3$ holds (in general $\dim(W) + \dim(W^\perp) = \dim(V)$).

**Theorem**. If $W$ is the span of the columns $\vec{u}_1, \ldots, \vec{u}_n$ of $m \times n$ matrix $A$ (the column space of $A$), then

$$W^\perp = \text{nullspace}(A^T) = \text{span}\{\text{Strang's Special Solutions for } A^T\vec{u} = \vec{0}\}.$$

**Proof**. Given $W = \text{span}\{\vec{u}_1, \ldots, \vec{u}_n\}$, then

$$\begin{aligned} W^\perp &= \{\vec{v} \,:\, \vec{v} \cdot \vec{w} = 0, \quad \text{all} \quad \vec{w} \in W\} \\ &= \{\vec{v} \,:\, \vec{u}_j \cdot \vec{v} = 0, \quad j = 1, \ldots, n\} \\ &= \{\vec{v} \,:\, A^T\vec{v} = \vec{0}\}. \end{aligned}$$

Strang's Special solutions are a basis for the homogeneous problem $A^T\vec{u} = \vec{0}$. Therefore, $W^\perp = \text{nullspace}(A^T) = \text{span}\{\text{Strang's Special Solutions for } A^T\vec{u} = \vec{0}\}.$

## Column Space, Row Space and Null Space of a Matrix $A$

The column space, row space and null space of an $m \times n$ matrix $A$ are sets in $\boldsymbol{R}^n$ or $\boldsymbol{R}^m$, defined to be the **span** of a certain set of vectors. The **span theorem** implies that each of these three sets are subspaces.

**Definition**. The **Column Space** of a matrix $A$ is the span of the columns of $A$, a subspace of $\boldsymbol{R}^m$. The **Pivot Theorem** implies that

$$\mathbf{colspace}(A) = \mathbf{span}\{\text{pivot columns of } A\}.$$

**Definition**. The **Row Space** of a matrix $A$ is the span of the rows of $A$, a subspace of $\boldsymbol{R}^n$. The definition implies two possible bases for this subspace, just one selected in an application:

$$\mathbf{rowspace}(A) = \mathbf{span}\{\text{Nonzero rows of rref}(A)\} = \mathbf{span}\{\text{pivot columns of } A^T\}.$$

**Definition**. The **Null Space** of a matrix $A$ is the set of all solutions $\vec{\mathrm{x}}$ to the homogeneous problem $A\vec{\mathrm{x}} = \vec{0}$, a subspace of $\boldsymbol{R}^n$. Because solution $\vec{\mathrm{x}}$ of $A\vec{\mathrm{x}} = \vec{0}$ is a linear combination of Strang's special solutions, then

$$\mathbf{nullspace}(A) = \mathbf{span}\{\text{Strang's Special Solutions for } A\vec{\mathrm{x}} = \vec{0}\}.$$

## The Row space is orthogonal to the Null Space

**Theorem**. Each row vector $\vec{\mathbf{r}}$ in matrix $\boldsymbol{A}$ satisfies $\vec{\mathbf{r}} \cdot \vec{\mathbf{x}} = \mathbf{0}$, where $\vec{\mathbf{x}}$ is a solution of the homogeneous equation $\boldsymbol{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Therefore

$$\mathbf{rowspace}(\boldsymbol{A}) \perp \mathbf{nullspace}(\boldsymbol{A}).$$

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The theorem is remembered from this diagram:

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \vec{\mathbf{x}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \quad \text{is equivalent to} \quad \begin{pmatrix} \mathbf{row}\ 1 \cdot \vec{\mathbf{x}} \\ \mathbf{row}\ 2 \cdot \vec{\mathbf{x}} \\ \mathbf{row}\ 3 \cdot \vec{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

which says that the rows of $\boldsymbol{A}$ are orthogonal to solutions $\vec{\mathbf{x}}$ of $\boldsymbol{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

## Computing the Orthogonal Complement of a subspace $W$

**Theorem**. In case $\boldsymbol{W}$ is the subspace of $\boldsymbol{R}^3$ spanned by two independent vectors $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$, then the orthogonal complement of $\boldsymbol{W}$ is the line through the origin generated by the cross product vector $\vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_2$:

$$\boldsymbol{W}^\perp = \mathbf{span}\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}^\perp = \mathbf{span}\{\vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_2\}$$

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**Theorem**. In case $\boldsymbol{W}$ is a subspace of $\boldsymbol{R}^m$ spanned by all column vectors $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_n$ of an $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$, then the orthogonal complement of $\boldsymbol{W}$ is the subspace

$$\begin{aligned} \boldsymbol{W}^\perp &= \mathbf{span}\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_n\}^\perp \\ &= \{\,\vec{\mathbf{y}} \,:\, \vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i = 0 \text{ for all } \boldsymbol{i} = 1, \ldots, \boldsymbol{n}\} \\ &= \mathbf{nullspace}(\boldsymbol{A}^T) \\ &= \mathbf{span}\{\text{Strang's Special Solutions for } A^T\vec{\mathrm{u}} = \vec{0}\} \end{aligned}$$

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**Method**. To compute a basis for $\boldsymbol{W}^\perp$, find Strang's special Solutions for the homogeneous problem $\boldsymbol{A}^T\vec{\mathbf{u}} = \vec{0}$. The basis size is $\boldsymbol{k} =$ number of free variables in $\boldsymbol{A}^T\vec{\mathbf{u}} = \vec{0}$.

Applications may add an additional step to replace this basis by the Gram-Schmidt orthogonal basis $\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_k$. Then $\boldsymbol{W}^\perp = \mathbf{span}\{\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_k\}$.

## Fundamental Theorem of Linear Algebra

**Definition**. The four fundamental subspaces are $\mathbf{rowspace}(\boldsymbol{A})$, $\mathbf{colspace}(\boldsymbol{A})$, $\mathbf{nullspace}(\boldsymbol{A})$ and $\mathbf{nullspace}(\boldsymbol{A}^T)$.

The **Fundamental Theorem of Linear Algebra** has two parts:

**(1)** Dimension of the Four Fundamental Subspaces.

Assume matrix $\boldsymbol{A}$ is $\boldsymbol{m} \times \boldsymbol{n}$ with $\boldsymbol{r}$ pivots. Then

$$\begin{aligned} &\mathbf{dim}(\mathbf{rowspace}(\boldsymbol{A})) = \boldsymbol{r}, &&\mathbf{dim}(\mathbf{colspace}(\boldsymbol{A})) = \boldsymbol{r}, \\ &\mathbf{dim}(\mathbf{nullspace}(\boldsymbol{A})) = \boldsymbol{n} - \boldsymbol{r}, &&\mathbf{dim}(\mathbf{nullspace}(\boldsymbol{A}^T)) = \boldsymbol{m} - \boldsymbol{r} \end{aligned}$$

**(2)** Orthogonality of the Four Fundamental Subspaces.

$$\begin{aligned} &\mathbf{rowspace}(\boldsymbol{A}) \perp \mathbf{nullspace}(\boldsymbol{A}) \\ &\mathbf{colspace}(\boldsymbol{A}) \perp \mathbf{nullspace}(\boldsymbol{A}^T) \end{aligned}$$

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Gilbert Strang's textbook *Linear Algebra* has a cover illustration for the fundamental theorem of linear algebra. The original article is *The Fundamental Theorem of Linear Algebra*, `http://www.jstor.org/stable/2324660`. The free 1993 jstor PDF is available via the Marriott library. Requires UofU 2-factor login.