Fundamental Theorem of Linear Algebra

- Orthogonal Vectors
- Orthogonal and Orthonormal Set
- Orthogonal Complement of a Subspace $W$
- Column Space, Row Space and Null Space of a Matrix $A$
- The **Fundamental Theorem of Linear Algebra**
Orthogonality

Definition 1 (Orthogonal Vectors)
Two vectors $\mathbf{u}, \mathbf{v}$ are said to be orthogonal provided their dot product is zero:

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

If both vectors are nonzero (not required in the definition), then the angle $\theta$ between the two vectors is determined by

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = 0,$$

which implies $\theta = 90^\circ$. In short, orthogonal vectors form a right angle.
Definition 2 (Orthogonal Set of Vectors)
A given set of nonzero vectors $\vec{u}_1, \ldots, \vec{u}_k$ that satisfies the orthogonality condition
$$\vec{u}_i \cdot \vec{u}_j = 0, \quad i \neq j,$$
is called an orthogonal set.

Definition 3 (Orthonormal Set of Vectors)
A given set of unit vectors $\vec{u}_1, \ldots, \vec{u}_k$ that satisfies the orthogonality condition is called an orthonormal set.
Orthogonal Complement $W^\perp$ of a Subspace $W$

**Definition.** Let $W$ be a subspace of an inner product space $V$, inner product $\langle \vec{u}, \vec{v} \rangle$. The **orthogonal complement** of $W$, denoted $W^\perp$, is the set of all vectors $\vec{v}$ in $V$ such that $\langle \vec{u}, \vec{v} \rangle = 0$ for all $\vec{u}$ in $W$. In set notation:

$$W^\perp = \{ \vec{v} : \langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \text{ in } W \}$$

**Example.** If $V = \mathbb{R}^3$ and $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$, then $W^\perp$ is the span of the calculus/physics cross product $\vec{u}_1 \times \vec{u}_2$. The equation $\dim(W) + \dim(W^\perp) = 3$ holds (in general $\dim(W) + \dim(W^\perp) = \dim(V)$).

**Theorem.** If $W$ is the span of the columns $\vec{u}_1, \ldots, \vec{u}_n$ of $m \times n$ matrix $A$ (the column space of $A$), then

$$W^\perp = \text{nullspace}(A^T) = \text{span}\{\text{Strang's Special Solutions for } A^T\vec{u} = \vec{0}\}.$$ 

**Proof.** Given $W = \text{span}\{\vec{u}_1, \ldots, \vec{u}_n\}$, then

$$W^\perp = \{ \vec{v} : \vec{v} \cdot \vec{w} = 0, \text{ all } \vec{w} \in W \}$$

$$= \{ \vec{v} : \vec{u}_j \cdot \vec{v} = 0, \text{ } j = 1, \ldots, n \}$$

$$= \{ \vec{v} : A^T\vec{v} = \vec{0} \}.$$ 

Strang’s Special solutions are a basis for the homogeneous problem $A^T\vec{u} = \vec{0}$. Therefore, $W^\perp = \text{nullspace}(A^T) = \text{span}\{\text{Strang's Special Solutions for } A^T\vec{u} = \vec{0}\}$. 
Column Space, Row Space and Null Space of a Matrix $A$

The column space, row space and null space of an $m \times n$ matrix $A$ are sets in $\mathbb{R}^n$ or $\mathbb{R}^m$, defined to be the span of a certain set of vectors. The span theorem implies that each of these three sets are subspaces.

**Definition.** The **Column Space** of a matrix $A$ is the span of the columns of $A$, a subspace of $\mathbb{R}^m$. The **Pivot Theorem** implies that

$$\text{colspace}(A) = \text{span}\{\text{pivot columns of } A\}.$$  

**Definition.** The **Row Space** of a matrix $A$ is the span of the rows of $A$, a subspace of $\mathbb{R}^n$. The definition implies two possible bases for this subspace, just one selected in an application:

$$\text{rowspace}(A) = \text{span}\{\text{Nonzero rows of } \text{rref}(A)\} = \text{span}\{\text{pivot columns of } A^T\}.$$  

**Definition.** The **Null Space** of a matrix $A$ is the set of all solutions $\vec{x}$ to the homogeneous problem $A\vec{x} = \vec{0}$, a subspace of $\mathbb{R}^n$. Because solution $\vec{x}$ of $A\vec{x} = \vec{0}$ is a linear combination of Strang’s special solutions, then

$$\text{nullspace}(A) = \text{span}\{\text{Strang’s Special Solutions for } A\vec{x} = \vec{0}\}.$$
The Row space is orthogonal to the Null Space

Theorem. Each row vector \( \vec{r} \) in matrix \( A \) satisfies \( \vec{r} \cdot \vec{x} = 0 \), where \( \vec{x} \) is a solution of the homogeneous equation \( A\vec{x} = \vec{0} \). Therefore

\[
\text{rowspace}(A) \perp \text{nullspace}(A).
\]

The theorem is remembered from this diagram:

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\quad \text{is equivalent to} \quad
\begin{pmatrix}
\text{row} 1 \cdot \vec{x} \\
\text{row} 2 \cdot \vec{x} \\
\text{row} 3 \cdot \vec{x} \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\]

which says that the rows of \( A \) are orthogonal to solutions \( \vec{x} \) of \( A\vec{x} = \vec{0} \).
Computing the Orthogonal Complement of a subspace $W$  

**Theorem.** In case $W$ is the subspace of $\mathbb{R}^3$ spanned by two independent vectors $\vec{u}_1, \vec{u}_2$, then the orthogonal complement of $W$ is the line through the origin generated by the cross product vector $\vec{u}_1 \times \vec{u}_2$:

$$W^\perp = \text{span}\{\vec{u}_1, \vec{u}_2\}^\perp = \text{span}\{\vec{u}_1 \times \vec{u}_2\}$$

**Theorem.** In case $W$ is a subspace of $\mathbb{R}^m$ spanned by all column vectors $\vec{u}_1, \ldots, \vec{u}_n$ of an $m \times n$ matrix $A$, then the orthogonal complement of $W$ is the subspace

$$W^\perp = \text{span}\{\vec{u}_1, \ldots, \vec{u}_n\}^\perp = \{ \vec{y} : \vec{y} \cdot \vec{u}_i = 0 \text{ for all } i = 1, \ldots, n \} = \text{nullspace}(A^T) = \text{span}\{\text{Strang’s Special Solutions for } A^T\vec{u} = \vec{0}\}$$

**Method.** To compute a basis for $W^\perp$, find Strang’s special Solutions for the homogeneous problem $A^T\vec{u} = \vec{0}$. The basis size is $k = \text{number of free variables in } A^T\vec{u} = \vec{0}$. Applications may add an additional step to replace this basis by the Gram-Schmidt orthogonal basis $\vec{y}_1, \ldots, \vec{y}_k$. Then $W^\perp = \text{span}\{\vec{y}_1, \ldots, \vec{y}_k\}$. 
**Fundamental Theorem of Linear Algebra**

**Definition.** The four fundamental subspaces are $\text{rowspace}(A)$, $\text{colspace}(A)$, $\text{nullspace}(A)$ and $\text{nullspace}(A^T)$.

The **Fundamental Theorem of Linear Algebra** has two parts:
1. **Dimension of the Four Fundamental Subspaces.**
   
   Assume matrix $A$ is $m \times n$ with $r$ pivots. Then
   
   $$\dim(\text{rowspace}(A)) = r, \quad \dim(\text{colspace}(A)) = r,$$
   $$\dim(\text{nullspace}(A)) = n - r, \quad \dim(\text{nullspace}(A^T)) = m - r$$

2. **Orthogonality of the Four Fundamental Subspaces.**

   $$\text{rowspace}(A) \perp \text{nullspace}(A)$$
   $$\text{colspace}(A) \perp \text{nullspace}(A^T)$$