

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

- a. A is an invertible matrix.
- b. A is row equivalent to the $n \times n$ identity matrix.
- c. A has n pivot positions.
- d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- e. The columns of A form a linearly independent set.
- f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
- h. The columns of A span \mathbb{R}^n .
- i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- j. There is an $n \times n$ matrix C such that $CA = I$.
- k. There is an $n \times n$ matrix D such that $AD = I$.
- l. A^T is an invertible matrix.

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THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

- m. The columns of A form a basis of \mathbb{R}^n .
- n. $\text{Col } A = \mathbb{R}^n$
- o. $\dim \text{Col } A = n$
- p. $\text{rank } A = n$
- q. $\text{Nul } A = \{\mathbf{0}\}$
- r. $\dim \text{Nul } A = 0$

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THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- s. The number 0 is *not* an eigenvalue of A .
- t. The determinant of A is *not* zero.

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THEOREM

The Invertible Matrix Theorem (concluded)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

- u. $(\text{Col } A)^\perp = \{\mathbf{0}\}$.
- v. $(\text{Nul } A)^\perp = \mathbb{R}^n$.
- w. $\text{Row } A = \mathbb{R}^n$.
- x. A has n nonzero singular values.

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