8.1 Eigenanalysis Applications

Discrete Dynamical Systems

The matrix equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}\vec{\mathbf{x}} \tag{1}$$

predicts the state $\vec{\mathbf{y}}$ of a system initially in state $\vec{\mathbf{x}}$ after some fixed elapsed time. The 3×3 matrix A in (1) represents the **dynamics** which changes state $\vec{\mathbf{x}}$ into state $\vec{\mathbf{y}}$. An equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ like equation (1) is called a **discrete dynamical system**. The fixed elapsed time for changing $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$ is called the **period** of the discrete dynamical system. Matrix A is called a **transition matrix**, provided A has nonnegative entries and column sums equal to one (see **Stochastic Matrices** below).

The eigenpairs of A in (1) are shown in *details* page 5 to be $(1, \vec{\mathbf{v}}_1)$, $(1/2, \vec{\mathbf{v}}_2)$, $(1/5, \vec{\mathbf{v}}_3)$ where the eigenvectors are given by

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}. \tag{2}$$

Market Shares

A typical application of discrete dynamical systems is telephone long distance company market shares x_1, x_2, x_3, which are fractions of the total market for long distance service. If three companies provide all the services, then their market fractions add to one: $x_1 + x_2 + x_3 = 1$. The equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ gives the market shares of the three companies after a fixed time period, say one year. Then market shares after one, two and three years are given by the **iterates**

$$\begin{aligned} \vec{\mathbf{y}}_1 &= A\vec{\mathbf{x}}, \\ \vec{\mathbf{y}}_2 &= A^2\vec{\mathbf{x}}, \\ \vec{\mathbf{y}}_3 &= A^3\vec{\mathbf{x}}. \end{aligned}$$

Fourier's replacement model gives succinct and useful formulas for the iterates: if $\vec{\mathbf{x}} = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3$, then

$$\begin{aligned} \vec{\mathbf{y}}_1 &= A\vec{\mathbf{x}} &&= a_1\lambda_1\vec{\mathbf{v}}_1 + a_2\lambda_2\vec{\mathbf{v}}_2 + a_3\lambda_3\vec{\mathbf{v}}_3, \\ \vec{\mathbf{y}}_2 &= A^2\vec{\mathbf{x}} &&= a_1\lambda_1^2\vec{\mathbf{v}}_1 + a_2\lambda_2^2\vec{\mathbf{v}}_2 + a_3\lambda_3^2\vec{\mathbf{v}}_3, \\ \vec{\mathbf{y}}_3 &= A^3\vec{\mathbf{x}} &&= a_1\lambda_1^3\vec{\mathbf{v}}_1 + a_2\lambda_2^3\vec{\mathbf{v}}_2 + a_3\lambda_3^3\vec{\mathbf{v}}_3. \end{aligned}$$

The advantage of Fourier's model is that an iterate A^n is computed directly, without computing the powers before it. Because $\lambda_1 = 1$ and

$\lim_{n\to\infty} |\lambda_2|^n = \lim_{n\to\infty} |\lambda_3|^n = 0$, then for large n

$$\vec{\mathbf{y}}_n \approx a_1(1)\vec{\mathbf{v}}_1 + a_2(0)\vec{\mathbf{v}}_2 + a_3(0)\vec{\mathbf{v}}_3 = \begin{pmatrix} 12a_1 \\ 15a_1 \\ 13a_1 \end{pmatrix}.$$

The numbers a_1, a_2, a_3 are related to x_1, x_2, x_3 in the expansion $\vec{\mathbf{x}} = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3$ by the equations $12a_1 - a_2 - 4a_3 = x_1$, $15a_1 + 3a_3 = x_2$, $13a_1 + a_2 + a_3 = x_3$. Due to $x_1 + x_2 + x_3 = 1$, the value of a_1 is given by $a_1 = 1/40$. The three market shares after a long time period are therefore predicted to be 3/10, 3/8, 13/40. The reader should verify the market share identity $\frac{3}{10} + \frac{3}{8} + \frac{13}{40} = 1$.

Stochastic Matrices

The special matrix A in (1) is a **stochastic matrix**[1], defined by the properties

$$\sum_{i=1}^{n} a_{ij} = 1, \quad a_{kj} \geq 0, \quad k, j = 1, \ldots, n.$$

The definition is memorized by the phrase *each column sum is one.*

Leontief input-output models, popularized by 1973 Nobel Prize economist Wassily Leontief, are stochastic models. A typical model is $A = R^T$ where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ .2 & .3 & .5 \\ .4 & .4 & .2 \end{pmatrix}.$$

The **rows** of R add to one, therefore the **columns** of A add to one. Row 1 is the bank, Row 2 is Factory 1, Row 3 is Factory 2. Matrix R tracks the money as it is being passed back and forth between the factories and the bank.

Leslie Models in population biology are similar to stochastic models. They predict the next age group population size based upon the previous previous population size. A Leslie matrix for $n = 4$ looks like

$$A = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{pmatrix}.$$

Neither the row sums nor the column sums are one. However, some stochastic matrix results have analogs for Leslie matrices.

[1]Technically, a **right** stochastic matrix, which means columns add to one. A **left** stochastic matrix has rows adding to one.

Theorem 1 (Stochastic Matrix Properties)
Let A be a stochastic matrix. Then

- **(a)** If $\vec{\mathbf{x}}$ is a vector with $x_1 + \cdots + x_n = 1$, then $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ satisfies $y_1 + \cdots + y_n = 1$.
- **(b)** If the components of $\vec{\mathbf{v}}$ are all 1, then $A^T\vec{\mathbf{v}} = \vec{\mathbf{v}}$. Therefore, $(1, \vec{\mathbf{v}})$ is an eigenpair of A^T.
- **(c)** One root of the characteristic equation $\det(A - \lambda I) = 0$ is $\lambda = 1$. All other roots satisfy $|\lambda| \le 1$.

Proof of Stochastic Matrix Properties:
(a) $\sum_{i=1}^n y_i = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\right) x_j = \sum_{j=1}^n (1) x_j = 1$.

(b) Entry j of $A^T\vec{\mathbf{v}}$ is given by $\sum_{i=1}^n (a_{ij})(1) = \text{column sum} = 1$.

(c) The determinant rule $\det(B^T) = \det(B)$ applied to $B = A - \lambda I$ implies A and A^T have the same eigenvalues. Apply **(b)** to verify that A has eigenvalue 1. Any other root λ of $|A - \lambda I| = 0$ is also a root of $|A^T - \lambda I| = 0$ with corresponding eigenvector $\vec{\mathbf{x}}$ satisfying $A^T\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$. Because $\vec{\mathbf{x}} \ne \vec{\mathbf{0}}$, then $\vec{\mathbf{x}}$ has a component x_j with largest magnitude $|x_j| > 0$. Isolate index j across equation $\lambda\vec{\mathbf{x}} = A^T\vec{\mathbf{x}}$, then divide by $|x_j|$, to obtain $\lambda = \sum_{i=1}^n a_{ij}\frac{x_i}{x_j}$. Because $a_{ji} \ge 0$ and $0 \le \left|\frac{x_i}{x_j}\right| \le 1$, then $|\lambda| \le 1$, because

$$|\lambda| \le \sum_{i=1}^n a_{ij}\left|\frac{x_i}{x_j}\right| \le \sum_{i=1}^n (a_{ij})(1) = \text{column sum} = 1.$$

Definition 1
Notation $A > 0$ means all $a_{ij} > 0$. Notation $A \le B$ means $a_{ij} \le b_{ij}$, also written $B \ge A$.

Definition 2
Matrix $\mathbf{max_r}(A)$ (resp. $\mathbf{min_r}(A)$) is obtained from A by replacing each entry a_{ij} by the maximum (resp. minimum) element of row i. Symbol $\delta = \min_{i,j} a_{ij}$. Matrix $\mathcal{O}$ is the $n \times n$ matrix of all ones.

Theorem 2 (Perron-Frobenius: Positive Stochastic Matrix)
Let A be a stochastic matrix all of whose entries are strictly positive. Then

- **(a)** There exists an eigenpair $(1, \vec{\mathbf{w}})$ of A such that $\vec{\mathbf{w}}$ has nonnegative components and $\lim_{n\to\infty} A^n = \langle \vec{\mathbf{w}} | \vec{\mathbf{w}} | \cdots | \vec{\mathbf{w}} \rangle$.
- **(b)** If $(1, \vec{\mathbf{v}})$ is an eigenpair of A, then $\vec{\mathbf{v}} = c\vec{\mathbf{w}}$ for $c = \sum_{i=1}^n v_i$. Briefly, the eigenspace for $\lambda = 1$ has dimension one.
- **(c)** If $\lambda \ne 1$ is a real or complex eigenvalue of A, then $|\lambda| < 1$.
- **(d)** If $(\lambda, \vec{\mathbf{v}})$ is an eigenpair of A and $\vec{\mathbf{v}}$ has nonnegative components, then all components of $\vec{\mathbf{v}}$ are strictly positive, $\lambda = 1$ and $\vec{\mathbf{v}} = c\vec{\mathbf{w}}$ for some constant c.

Proof of the Perron-Frobenius Theorem:[2]

Proof of (a)

The proof is organized as five lemmas. Assume throughout that $A > 0$ is stochastic with least element δ, $B \geq 0$ and $\mathcal{O}$ is the matrix of all ones.

Lemma 1a. If A, B are stochastic, then BA is stochastic.

Lemma 2a. $\mathbf{min_r}(B) \leq \mathbf{min_r}(BA) \leq BA \leq \mathbf{max_r}(BA) \leq \mathbf{max_r}(B)$.

Proof: The maximum along row i of $C = BA$ is some $c_{ij} = \sum_{k=1}^n b_{ik}a_{kj}$. Let M denote the maximum along row i of B. Because columns of A sum to 1, then $c_{ij} = \sum_{k=1}^n b_{ik}a_{kj} \leq \sum_{k=1}^n M a_{kj} = M$. Then $BA \leq \mathbf{max_r}(BA) \leq \mathbf{max_r}(B)$. Details for inequality $\mathbf{min_r}(B) \leq \mathbf{min_r}(BA) \leq BA$ are similar.

Lemma 3a. $\mathbf{max_r}(BA) - \mathbf{min_r}(BA) \leq (1-\delta)\,(\mathbf{max_r}(B) - \mathbf{min_r}(B))$.

Proof: Let $C = BA$ have row i maximum at c_{ij} and row minimum at c_{ik}. Then all elements in row i of matrix $\mathbf{max_r}(BA) - \mathbf{min_r}(BA)$ have value $S = c_{ij} - c_{ik}$. Let M (resp. m) be the common entry along row i of $\mathbf{max_r}(B)$ (resp. $\mathbf{min_r}(B)$). We'll verify $S \leq (1-\delta)\,(M-m)$, which proves the lemma.

Re-write $S = c_{ij} - c_{ik} = \sum_{p=1}^n b_{ip}a_{pj} - \sum_{p=1}^n b_{ip}a_{pk} = \sum_{p=1}^n b_{ip}(a_{pj} - a_{pk})$. Let $p_1, \ldots, p_r$ be the set of indices p such that $a_{pj} - a_{pk} > 0$ and let $q_1, \ldots, q_s$ be the set of indices q such that $a_{qj} - a_{qk} < 0$. Indices p that satisfy $a_{pj} - a_{pk} = 0$ contribute zero to S. In cases $r = 0$ and/or $s = 0$ we have $S \leq 0$, so the conclusion follows. Henceforth, assume $r \geq 1$ and $s \geq 1$. The column sums of A are 1, which implies for instance $\sum_{\ell=1}^r a_{p_\ell j} + \sum_{\ell=1}^s a_{q_\ell j} = 1$. We estimate:

$$
\begin{aligned}
S &= \textstyle\sum_{p=1}^n b_{ip}(a_{pj} - a_{pk}) \\
&= \textstyle\sum_{\ell=1}^r b_{ip}(a_{p_\ell j} - a_{p_\ell k}) + \sum_{\ell=1}^s b_{ip}(a_{q_\ell j} - a_{q_\ell k}) \\
&\leq M \textstyle\sum_{\ell=1}^r (a_{p_\ell j} - a_{p_\ell k}) + m \sum_{\ell=1}^s (a_{q_\ell j} - a_{q_\ell k}) \\
&= M\,(1 - \textstyle\sum_{\ell=1}^s a_{q_\ell j} - 1 + \sum_{\ell=1}^s a_{q_\ell k}) + m \sum_{\ell=1}^s (a_{q_\ell j} - a_{q_\ell k}) \\
&= (M-m)\,(-\textstyle\sum_{\ell=1}^s a_{q_\ell j} + \sum_{\ell=1}^s a_{q_\ell k}) \\
&\leq (M-m)\,(-s\delta + 1) \\
&\leq (M-m)\,(-\delta + 1)\,.
\end{aligned}
$$

Lemma 4a. $\mathbf{max_r}(A^{k+1}) - \mathbf{min_r}(A^{k+1}) \leq (1-\delta)^k \mathcal{O}$.

Proof: Let $B = A^k$ and apply Lemmas 1a and 3a. Then $\mathbf{max_r}(A^{k+1}) - \mathbf{min_r}(A^{k+1}) \leq (1-\delta)\left(\mathbf{max_r}(A^k) - \mathbf{min_r}(A^k)\right)$. Induction on k implies the result, because $\mathbf{max_r}(A) - \mathbf{min_r}(A) \leq \mathcal{O}$.

Lemma 5a. There exists a vector $\vec{\mathbf{w}}$ with all positive components such that $\lim_{k\to\infty} A^k = \langle \vec{\mathbf{w}}\,|\vec{\mathbf{w}}\,|\cdots|\vec{\mathbf{w}}\,\rangle$. Then $A\vec{\mathbf{w}} = \vec{\mathbf{w}}$ and $(1, \vec{\mathbf{w}}\,)$ is an eigenpair.[3]

Proof: The preceding lemmas and the calculus squeeze theorem for limits imply that $\mathbf{max_r}(A^k)$ and $\mathbf{min_r}(A^k)$ converge as $k \to \infty$ to some matrix P. Because $\mathbf{max_r}(A^k)$ has identical elements in each row, then so does P. Therefore, the columns of P are the same vector $\vec{\mathbf{w}}$. Take limits across inequality $\mathbf{min_r}(A^k) \geq \delta\mathcal{O}$ to prove $\vec{\mathbf{w}} > \vec{\mathbf{0}}$. Vector $\vec{\mathbf{w}}$ equals $P\vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ = column 1 of the identity matrix. Then $\vec{\mathbf{w}} = P\vec{\mathbf{u}} = \lim_{k\to\infty} A^{k+1}\vec{\mathbf{u}} = A\left(\lim_{k\to\infty} A^k\vec{\mathbf{u}}\right) = A\vec{\mathbf{w}}$, which is the eigenpair equation $\vec{\mathbf{w}} = A\vec{\mathbf{w}}$.

Proof of (b)

Eigenpair equation $\vec{\mathbf{v}} = A\vec{\mathbf{v}}$ is multiplied repeatedly by A to give $\vec{\mathbf{v}} = A^{k+1}\vec{\mathbf{v}}$.

[2] Perron-Frobenius theory is a basis for the Google Search **PageRank** algorithm.

[3] The numerical **power method** can be used to approximate eigenvector $\vec{\mathbf{w}}$.

Take the limit using part (a): $\vec{\mathbf{v}} = P\vec{\mathbf{v}}$, where $P = \langle \vec{\mathbf{w}} | \vec{\mathbf{w}} | \cdots | \vec{\mathbf{w}} \rangle$. Then $\vec{\mathbf{v}} = P\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}}$.

Proof of (c)

Consider an eigenpair $(\lambda, \vec{\mathbf{v}})$. Apply A across $\lambda\vec{\mathbf{v}} = A\vec{\mathbf{v}}$ to obtain $\lambda^k\vec{\mathbf{v}} = A^k\vec{\mathbf{v}}$. Use part (a) to take the limit as $k \to \infty$. Then, as in part (b), $\lim_{k\to\infty} \lambda^k\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}}$. This limit exists only in case $|\lambda| \le 1$. If $|\lambda| = 1$, then $\lambda = e^{i\theta}$ for some angle θ. The limit fails to exist unless $\theta = 0$ modulo 2π. Therefore, $\lambda = 1$ and $\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}}$.

Proof of (d)

Let's suppose some $v_j = 0$, in order to reach a contradiction. Component j of the identity $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ says that $\sum_{k=1}^n a_{jk}v_k = 0$. Because $\vec{\mathbf{v}} \ne \vec{\mathbf{0}}$, then at least one $v_k \ne 0$. Because $a_{jk} > 0$, then $\sum_{k=1}^n a_{jk}v_k > 0$, a contradiction.

The proof of the Perron-Frobenius theorem is complete.

Details for the eigenpairs of (1): To be computed are the eigenvalues λ and eigenvectors $\vec{\mathbf{v}}$ for the 3×3 matrix

$$A = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}.$$

The eigenpairs are $(1, \vec{\mathbf{v}}_1), \left(\frac{1}{2}, \vec{\mathbf{v}}_2\right), \left(\frac{1}{5}, \vec{\mathbf{v}}_3\right)$ where

$$(3) \qquad \vec{\mathbf{v}}_1 = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}.$$

Eigenvalues. The roots $\lambda = 1, 1/2, 1/5$ of the characteristic equation $\det(A - \lambda I) = 0$ are found by these details:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} .5-\lambda & .4 & 0 \\ .3 & .5-\lambda & .3 \\ .2 & .1 & .7-\lambda \end{vmatrix} \\ &= \frac{1}{10} - \frac{8}{10}\lambda + \frac{17}{10}\lambda^2 - \lambda^3 && \text{Expand by cofactors.} \\ &= -\frac{1}{10}(\lambda-1)(2\lambda-1)(5\lambda-1) && \text{Factor the cubic.} \end{aligned}$$

The factorization was found by long division of the cubic by $\lambda - 1$, the idea born from the fact that 1 is a root and therefore $\lambda - 1$ is a factor, by the Factor Theorem of college algebra. The root $\lambda = 1$ was discovered from the Rational Root theorem of college algebra.[4]

Eigenpairs. To each eigenvalue $\lambda = 1, 1/2, 1/5$ corresponds one **rref** calculation, to find the eigenvectors paired to λ. The three eigenvectors are given by (2). The details:

Eigenvalue $\lambda = 1$.

[4]A rational root x of $a_nx^n + \cdots + a_0 = 0$ is a rational factor of a_0/a_n.

$$
\begin{aligned}
A-(1)I &= \begin{pmatrix} .5-1 & .4 & 0 \\ .3 & .5-1 & .3 \\ .2 & .1 & .7-1 \end{pmatrix} \\
&\approx \begin{pmatrix} -5 & 4 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Multiply rule, multiplier=10.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule twice.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 0 & 13 & -15 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \end{pmatrix} && \text{Multiply rule and combination rule.} \\
&\approx \begin{pmatrix} 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathbf{rref}(A-(1)I)
\end{aligned}
$$

An equivalent reduced echelon system is $x - 12z/13 = 0$, $y - 15z/13 = 0$. The free variable assignment is $z = t_1$ and then $x = 12t_1/13$, $y = 15t_1/13$.

An eigenvector can be selected as the partial derivative on variable t_1 across the general solution $x = 12t_1/13$, $y = 15t_1/13$, $z = t_1$ (equivalent here to setting $t_1 = 1$). This computation gives eigenvector $x = 12/13, y = 15/13, z = 1$.

An eigenvector can be multiplied by a constant $c \neq 0$ to obtain another eigenvector. To eliminate fractions in the answer, the practice is to multiply by an integer c to eliminate all fractions. Choose constant $c = 13$ to obtain eigenvector $x = 12, y = 15, z = 13$.

Eigenvalue $\lambda = 1/2$.

$$
\begin{aligned}
A-(1/2)I &= \begin{pmatrix} .5-.5 & .4 & 0 \\ .3 & .5-.5 & .3 \\ .2 & .1 & .7-.5 \end{pmatrix} \\
&\approx \begin{pmatrix} 0 & 4 & 0 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{pmatrix} && \text{Multiply rule, factor=10.} \\
&\approx \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply rules.} \\
&= \mathbf{rref}(A-.5I)
\end{aligned}
$$

An eigenvector is found from the equivalent reduced echelon system $y = 0$, $x + z = 0$ to be $x = -1$, $y = 0$, $z = 1$.

Eigenvalue $\lambda = 1/5$.

$$A - (1/5)I = \begin{pmatrix} .5-.2 & .4 & 0 \\ .3 & .5-.2 & .3 \\ .2 & .1 & .7-.2 \end{pmatrix}$$

$$\approx \begin{pmatrix} 3 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 5 \end{pmatrix} \quad \text{Multiply rule.}$$

$$\approx \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Combination rule.}$$

$$= \mathbf{rref}(A - (1/5)I)$$

An eigenvector is found from the equivalent reduced echelon system $x+4z=0$, $y-3z=0$ to be $x=-4$, $y=3$, $z=1$.

An answer check in `maple`:

```
with(LinearAlgebra):
A:=(1/10)*Matrix([[5,4,0],[3,5,3],[2,1,7]]);
B:=A-lambda*IdentityMatrix(3);
DD,P:=Eigenvectors(A);
factor(Determinant(B));
```

Coupled and Uncoupled Systems

The linear system of differential equations

$$\begin{aligned} x_1' &= -x_1 - x_3, \\ x_2' &= 4x_1 - x_2 - 3x_3, \\ x_3' &= 2x_1 - 4x_3, \end{aligned} \tag{4}$$

is called **coupled**, whereas the linear system of growth-decay equations

$$\begin{aligned} y_1' &= -3y_1, \\ y_2' &= -y_2, \\ y_3' &= -2y_3, \end{aligned} \tag{5}$$

is called **uncoupled**. The terminology *uncoupled* means that each differential equation in system (5) depends on exactly one variable, e.g., $y_1' = -3y_1$ depends only on variable y_1. In a *coupled* system, one of the differential equations must involve two or more variables.

Matrix Formulaton

Coupled system (4) and uncoupled system (5) can be written in matrix form, $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}' = D\vec{\mathbf{y}}$, with coefficient matrices

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If the coefficient matrix is **diagonal**, then the system is **uncoupled**. If the coefficient matrix is **not diagonal**, then one of the corresponding differential equations involves two or more variables and the system is called **coupled** or **cross-coupled**.

Solving Uncoupled Systems

An uncoupled system consists of independent growth-decay equations of the form $u' = au$. The solution formula $u = ce^{at}$ then leads to the general solution of the system of equations. For instance, system (5) has general solution

$$\begin{aligned} y_1 &= c_1 e^{-3t}, \\ y_2 &= c_2 e^{-t}, \\ y_3 &= c_3 e^{-2t}, \end{aligned} \tag{6}$$

where c_1, c_2, c_3 are **arbitrary constants**. The number of constants equals the dimension of the diagonal matrix D.

Coordinates and Coordinate Systems

If vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are independent in $\mathcal{R}^3$, then augmented matrix

$$P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle$$

is invertible. The columns $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ of P are called a **coordinate system**. The matrix P is called a **change of coordinates**.

Every vector $\vec{\mathbf{v}}$ in $\mathcal{R}^3$ can be uniquely expressed as

$$\vec{\mathbf{v}} = t_1 \vec{\mathbf{v}}_1 + t_2 \vec{\mathbf{v}}_2 + t_3 \vec{\mathbf{v}}_3.$$

The values t_1, t_2, t_3 are called the **coordinates** of $\vec{\mathbf{v}}$ relative to the basis $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, or more succinctly, the **coordinates of $\vec{\mathbf{v}}$ relative to** P.

Viewpoint of a Driver

The physical meaning of a coordinate system $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ can be understood by considering an auto going up a mountain road. Choose orthogonal $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ to give positions in the driver's seat and define $\vec{\mathbf{v}}_3$ be the seat-back direction. These are **local coordinates** as viewed from the driver's seat. The road map coordinates x, y and the altitude z define the **global coordinates** for the auto's position $\vec{\mathbf{p}} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

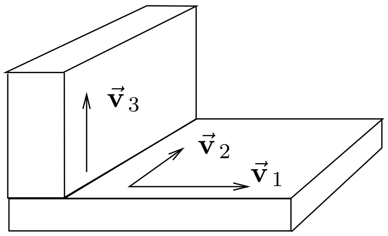


Figure 1. An auto seat.
The vectors $\vec{\mathbf{v}}_1(t)$, $\vec{\mathbf{v}}_2(t)$, $\vec{\mathbf{v}}_3(t)$ form an orthogonal triad which is a local coordinate system from the driver's viewpoint. The orthogonal triad changes continuously in t.

Change of Coordinates

A coordinate change from $\vec{\mathbf{y}}$ to $\vec{\mathbf{x}}$ is a linear algebraic equation $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ where the $n \times n$ matrix P is required to be invertible ($\det(P) \neq 0$). To illustrate, an instance of a change of coordinates from $\vec{\mathbf{y}}$ to $\vec{\mathbf{x}}$ is given by the linear equations

(7) $$\vec{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \vec{\mathbf{y}} \quad \text{or} \quad \begin{cases} x_1 &= y_1 + y_3, \\ x_2 &= y_1 + y_2 - y_3, \\ x_3 &= 2y_1 + y_3. \end{cases}$$

Constructing Coupled Systems

A general method exists to construct rich examples of coupled systems. The idea is to substitute a change of variables into a given uncoupled system. Consider a diagonal system $\vec{\mathbf{y}}\,' = D\vec{\mathbf{y}}$, like (5), and a change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$, like (7). Differential calculus applies to give

(8) $$\begin{aligned} \vec{\mathbf{x}}\,' &= (P\vec{\mathbf{y}})' \\ &= P\vec{\mathbf{y}}\,' \\ &= PD\vec{\mathbf{y}} \\ &= PDP^{-1}\vec{\mathbf{x}}. \end{aligned}$$

The matrix $A = PDP^{-1}$ is *not triangular* in general, and therefore the change of variables produces a **cross-coupled** system.

An illustration. To give an example, substitute into uncoupled system (5) the change of variable equations (7). Use equation (8) to obtain

(9) $$\vec{\mathbf{x}}\,' = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \vec{\mathbf{x}} \quad \text{or} \quad \begin{cases} x_1' = -x_1 - x_3, \\ x_2' = 4x_1 - x_2 - 3x_3, \\ x_3' = 2x_1 - 4x_3. \end{cases}$$

This **cross-coupled** system (9) can be solved using relations (7), (6) and $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ to give the general solution

(10) $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{-t} \\ c_3 e^{-2t} \end{pmatrix}.$$

Changing Coupled Systems to Uncoupled

We ask this question, motivated by the above calculations:

> Can every coupled system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ be subjected to a change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ which converts the system into a completely uncoupled system for variable $\vec{\mathbf{y}}(t)$?

Under certain circumstances, this is true, and it leads to an elegant and especially simple expression for the general solution of the differential system, as in (10):

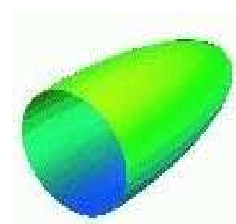
$$\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t).$$

The **task of eigenanalysis** is to simultaneously calculate from a cross-coupled system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ the change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ and the diagonal matrix D in the uncoupled system $\vec{\mathbf{y}}' = D\vec{\mathbf{y}}$

The **eigenanalysis coordinate system** is the set of n independent vectors extracted from the columns of P. In this coordinate system, the cross-coupled differential system (4) simplifies into a system of uncoupled growth-decay equations (5). Hence the terminology, *the method of simplifying coordinates.*

Eigenanalysis and Footballs

An ellipsoid or *football* is a geometric object described by its **semi-axes** (see Figure 2). In the vector representation, the **semi-axis directions** are unit vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the **semi-axis lengths** are the constants a, b, c. The vectors $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ form an **orthogonal triad**.



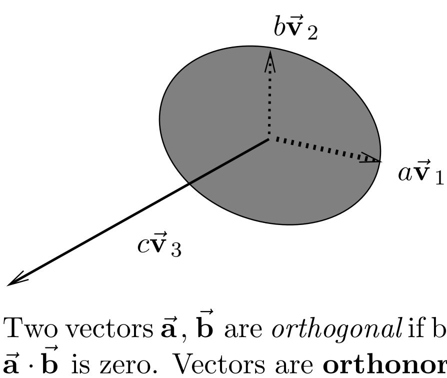


Figure 2. USA football. An ellipsoid is built from orthonormal semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the semi-axis lengths a, b, c. The semi-axis vectors are $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$.

Two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ are *orthogonal* if both are nonzero and their dot product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is zero. Vectors are **orthonormal** if each has unit length and they are pairwise orthogonal. The orthogonal triad $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ is an **invariant** of the ellipsoid's algebraic representations. Algebra does not change the triad: the invariants $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ must somehow be **hidden** in the equations that represent the football.

Algebraic eigenanalysis finds the hidden invariant triad $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ from the ellipsoid's algebraic equations. Suppose, for instance, that the equation of the ellipsoid is supplied as

$$x^2 + 4y^2 + xy + 4z^2 = 16.$$

A symmetric matrix A is constructed in order to write the equation in the form $\vec{\mathbf{X}}^T A \vec{\mathbf{X}} = 16$, where $\vec{\mathbf{X}}$ has components x, y, z. The replacement equation is[5]

$$(11) \qquad \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 16.$$

It is the 3×3 symmetric matrix A in (11) that is subjected to algebraic eigenanalysis. The matrix calculation will compute the unit semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, called the **hidden vectors** or **eigenvectors**. The semi-axis lengths a, b, c are computed at the same time, by finding the **hidden values**[6] or **eigenvalues** λ_1, λ_2, λ_3, known to satisfy the relations

$$\lambda_1 = \frac{16}{a^2}, \quad \lambda_2 = \frac{16}{b^2}, \quad \lambda_3 = \frac{16}{c^2}.$$

For the illustration, the football dimensions are $a = 2$, $b = 1.98$, $c = 4.17$. Details of the computation are delayed until page 13.

The Ellipse and Eigenanalysis

An ellipse equation in **standard form** is $\lambda_1 u^2 + \lambda_2 v^2 = 1$, where $\lambda_1 = 1/a^2$, $\lambda_2 = 1/b^2$ are expressed in terms of the semi-axis lengths a, b. The expression $\lambda_1 u^2 + \lambda_2 v^2$ is called a **quadratic form**. The study of the ellipse $\lambda_1 u^2 + \lambda_2 v^2 = 1$ is equivalent to the study of the quadratic form equation

$$\vec{\mathbf{r}}^T D \vec{\mathbf{r}} = 1, \quad \text{where} \quad \vec{\mathbf{r}} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Cross-terms. An ellipse may be represented by an equation in a xy-coordinate system having a cross-term xy, e.g., $4x^2 + 8xy + 10y^2 = 5$. The expression $4x^2 + 8xy + 10y^2$ is again called a quadratic form. Calculus courses provide methods to eliminate the cross-term and represent the equation in standard form, by a **rotation** by angle θ of the xy-system into the uv-system:

$$\begin{pmatrix} u \\ v \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

[5]The reader should pause here and multiply matrices in order to verify this statement. Halving of the entries corresponding to cross-terms generalizes to any ellipsoid.

[6]The terminology *hidden* arises because neither the semi-axis lengths nor the semi-axis directions are revealed directly by the ellipsoid equation.

Eigenanalysis computes angle θ through the columns of R, which are the unit semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ for the ellipse $4x^2 + 8xy + 10y^2 = 5$. If the quadratic form $4x^2 + 8xy + 10y^2$ is represented as $\vec{\mathbf{r}}^T A \vec{\mathbf{r}}$, then

$$\vec{\mathbf{r}} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}, \quad R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

$$\lambda_1 = 12, \quad \vec{\mathbf{v}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 2, \quad \vec{\mathbf{v}}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Ellipse equations. There are two coordinate systems, the xy-system and the rotated uv-system. The equations in each system, each divided by 5:

$$\text{(12)} \qquad \begin{aligned} \tfrac{4}{5}x^2 + \tfrac{8}{5}xy + 2y^2 &= 1, \\ \tfrac{2}{5}u^2 + \tfrac{12}{5}v^2 &= 1. \end{aligned}$$

The rotation relation $\begin{pmatrix} u \\ v \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$ is the set of equations

$$\text{(13)} \qquad \begin{cases} u &= \;= \frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y, \\ v &= \;= \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y, \end{cases}$$

which upon substitution into the uv-equation in (12) gives

$$\frac{2}{5}\left(\frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y\right)^2 + \frac{12}{5}\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right)^2 = 1.$$

The reader can verify that this is the first equation in (12).

Rotation matrix angle θ. The components of unit eigenvector $\vec{\mathbf{v}}_1$ can be used to determine $\theta = -63.4°$:

$$\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{cases} \cos\theta &= \frac{1}{\sqrt{5}}, \\ -\sin\theta &= \frac{2}{\sqrt{5}}. \end{cases}$$

The interpretation of angle θ: rotate the orthonormal basis $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ by angle $\theta = -63.4°$ in order to obtain the standard unit basis vectors $\vec{\imath}$, $\vec{\jmath}$. Most calculus texts discuss only the inverse rotation, where x, y are given in terms of u, v. In these references, θ is the negative of the value given here, due to a different geometric viewpoint.[7]

Semi-axis lengths. The lengths $a \approx 1.55$, $b \approx 0.63$ for the ellipse $4x^2 + 8xy + 10y^2 = 5$ are computed from the eigenvalues $\lambda_1 = 12$, $\lambda_2 = 2$ of matrix A by the equations

$$\frac{\lambda_1}{5} = \frac{1}{a^2}, \quad \frac{\lambda_2}{5} = \frac{1}{b^2}.$$

[7]Rod Serling, author and playwright for *The Twilight Zone*, enjoyed the view from the other side.

Geometry. The ellipse $4x^2 + 8xy + 10y^2 = 5$ is completely determined by the orthogonal semi-axis vectors $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$. The rotation R is a rigid motion mapping xy-plane vectors $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$ into uv-plane vectors $a\vec{\imath}$, $b\vec{\jmath}$. The θ-rotation R maps $4x^2+8xy+10y^2 = 5$ into the uv-equation $\lambda_1 u^2 + \lambda_2 v^2 = 5$, where λ_1, λ_2 are the eigenvalues of A. To see why, let $\vec{\mathbf{r}} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\vec{\mathbf{s}} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the equation $\vec{\mathbf{r}} = R\vec{\mathbf{s}}$. Then $\vec{\mathbf{r}}^T A\vec{\mathbf{r}} = \vec{\mathbf{s}}^T(R^T AR)\vec{\mathbf{s}}$. Using $R^T R = I$ gives $R^{-1} = R^T$ and $R^T AR = \mathbf{diag}(\lambda_1, \lambda_2)$. Finally, $\vec{\mathbf{r}}^T A\vec{\mathbf{r}} = \lambda_1 u^2 + \lambda_2 v^2$.

Orthogonal Triad Computation

Let's compute the semiaxis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ for the ellipsoid $x^2 + 4y^2 + xy + 4z^2 = 16$. To be applied is Theorem **??**. As explained on page 11, the starting point is to represent the ellipsoid equation as a quadratic form $\vec{\mathbf{W}}^T A\vec{\mathbf{W}} = 16$, where the symmetric matrix A and vector $\vec{\mathbf{W}}$ are defined by

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \vec{\mathbf{W}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

College algebra. The **characteristic polynomial** $\det(A - \lambda I) = 0$ determines the eigenvalues or hidden values of the matrix A. By cofactor expansion, this polynomial equation is

$$(4-\lambda)((1-\lambda)(4-\lambda) - 1/4) = 0$$

with roots 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

Eigenpairs. It will be shown that three eigenpairs are

$$\lambda_1 = 4, \quad \vec{\mathbf{x}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_2 = \frac{5+\sqrt{10}}{2}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{10}-3 \\ 1 \\ 0 \end{pmatrix},$$

$$\lambda_3 = \frac{5-\sqrt{10}}{2}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} \sqrt{10}+3 \\ -1 \\ 0 \end{pmatrix}.$$

The vector norms of the eigenvectors are given by $\|\vec{\mathbf{x}}_1\| = 1$, $\|\vec{\mathbf{x}}_2\| = \sqrt{20+6\sqrt{10}}$, $\|\vec{\mathbf{x}}_3\| = \sqrt{20-6\sqrt{10}}$. The orthonormal semi-axis directions $\vec{\mathbf{v}}_k = \vec{\mathbf{x}}_k/\|\vec{\mathbf{x}}_k\|$, $k = 1, 2, 3$, are then given by the formulas

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} \frac{\sqrt{10}-3}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20-6\sqrt{10}}} \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} \frac{\sqrt{10}+3}{\sqrt{20+6\sqrt{10}}} \\ \frac{-1}{\sqrt{20+6\sqrt{10}}} \\ 0 \end{pmatrix}.$$

Eigenpair Details.

$$\langle A - \lambda_1 I, \vec{\mathbf{0}} \rangle = \left(\begin{array}{ccc|c} 1-4 & 1/2 & 0 & 0 \\ 1/2 & 4-4 & 0 & 0 \\ 0 & 0 & 4-4 & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{Used Toolkit rules combination, multiply and swap. Found } \textbf{rref}.$$

$$\langle A - \lambda_2 I, \vec{\mathbf{0}} \rangle = \left(\begin{array}{ccc|c} \frac{-3-\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3-\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3-\sqrt{10}}{2} & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 3-\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{Toolkit rules applied. Found } \textbf{rref}.$$

$$\langle A - \lambda_3 I, \vec{\mathbf{0}} \rangle = \left(\begin{array}{ccc|c} \frac{-3+\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3+\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{10}}{2} & 0 \end{array}\right)$$

$$\approx \left(\begin{array}{ccc|c} 1 & 3+\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{Toolkit rules applied. Found } \textbf{rref}.$$

Solving the corresponding reduced echelon systems gives the preceding formulas for the eigenvectors $\vec{\mathbf{x}}_1$, $\vec{\mathbf{x}}_2$, $\vec{\mathbf{x}}_3$. The equation for the ellipsoid is $\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 16$, where the multipliers of the square terms are the eigenvalues of A and X, Y, Z define the new coordinate system determined by the eigenvectors of A. This equation can be re-written in the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1$, provided the semi-axis lengths a, b, c are defined by the relations $a^2 = 16/\lambda_1$, $b^2 = 16/\lambda_2$, $c^2 = 16/\lambda_3$. After computation, $a = 2$, $b = 1.98$, $c = 4.17$.

Exercises 8.1

Discrete Dynamical Systems

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 1 & 0 \\ 3 & 4 & 3 \\ 2 & 5 & 7 \end{pmatrix} \vec{\mathbf{x}} \tag{14}$$

1. Find eigenpair packages of A. Answers:
$D=\begin{pmatrix} .5 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P=\begin{pmatrix} -1 & 1 & 1 \\ 0 & -4 & 5 \\ 1 & 3 & 9 \end{pmatrix}$

2. Explain: A is a **transition matrix**.[8]

3. Assume $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ has period one year. Find the system state after two years.

4. Explain: $A^n\vec{\mathbf{x}}$ is the system state after n periods.

Market Shares

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix} \vec{\mathbf{x}} \tag{15}$$

5. Verify the eigenpairs of A using software.

6. Compute A^2, A^3, A^4 using software. Predict the limit of A^n as n approaches infinity.

7. Compute with software (rounded)

$$A^{10}=\begin{pmatrix} .30 & .30 & .30 \\ .37 & .38 & .37 \\ .32 & .32 & .33 \end{pmatrix}. \tag{16}$$

8. Let $\vec{\mathbf{x}}=\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Compute
$A^{10}\vec{\mathbf{x}} = \begin{pmatrix} 0.30 \\ 0.37 \\ 0.33 \end{pmatrix}$ (rounded)
in two ways by calculator:
(1) Fourier Replacement.
(2) Matrix multiply using (16).

Stochastic Matrices

9. Establish the identity $|A - \lambda I| = |A^T - \lambda I$.

10. Explain why A and A^T have the same eigenvalues but not necessarily the same eigenvectors.

11. Verify $\mathbf{max_r}(A) = \langle \vec{\mathbf{w}} | \vec{\mathbf{w}} | \cdots | \vec{\mathbf{w}} \rangle$, where $\vec{\mathbf{w}}$ has components $w_i = \max\{a_{ij}, 1 \le j \le n\}$.

12. Verify $\mathbf{max_r}(A) = D\mathcal{O}$, where D is the diagonal matrix of row maxima and $\mathcal{O}$ is the matrix of all ones.

Perron-Frobenius Theorem

Let $A > 0$ be $n \times n$ stochastic with unique eigenpair $(1, \vec{\mathbf{w}})$, all $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. Assume $\vec{\mathbf{v}} \ge \vec{\mathbf{0}}$, $\sum_{i=1}^n v_i = 1$ and $\delta = \min_{i,j} a_{ij}$.

13. Apply inequality $\mathbf{min_r}(A^n)\vec{\mathbf{v}} \le A^n\vec{\mathbf{v}} \le \mathbf{max_r}(A^n)\vec{\mathbf{v}}$ to prove $\lim_{n\to\infty} A^n\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}}$.

14. Verify Euclidean norm inequality $\|A^{k+1}\vec{\mathbf{v}} - \vec{\mathbf{w}}\| \le \sqrt{n}\,(1-\delta)^k$

Weierstrass Proof

These exercises establish existence of an eigenpair $(1, \vec{\mathbf{v}})$ for stochastic A having only nonnegative entries.

Weierstrass Compactness Theorem
A sequence of vectors $\{\vec{\mathbf{v}}_i\}_{i=1}^\infty$ contained in a closed, bounded set K in $\mathcal{R}^n$ has a subsequence converging in the vector norm of $\mathcal{R}^n$ to some vector $\vec{\mathbf{v}}$ in K.

Define set K to be all vectors $\vec{\mathbf{v}}$ with nonnegative components adding to 1. Let $\vec{\mathbf{v}}_0$ be any element of K. Assume stochastic A with $a_{ij} \ge 0$ and define $\vec{\mathbf{v}}_N = \frac{1}{N}\sum_{j=0}^{N-1} A^j\vec{\mathbf{v}}_0$.

15. Verify K is closed and bounded in $\mathcal{R}^n$. Then prove $\lambda\vec{\mathbf{x}} + (1-\lambda)\vec{\mathbf{y}}$ is in K for $0 \le \lambda \le 1$ and $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in K.

[8] Perron-Frobenius theory extensions in the literature apply to transition matrices. See the Weierstrass Proof exercises.

16. Prove identity
$\vec{\mathbf{v}}_{N+1} = \lambda \vec{\mathbf{v}}_N + (1-\lambda)A^N \vec{\mathbf{v}}_0$
where $\lambda = \frac{N}{N+1}$ and then prove by induction that $\vec{\mathbf{v}}_N$ is in K.

17. Verify all hypotheses in the Weierstrass theorem applied to $\{\vec{\mathbf{v}}_N\}_{N=0}^{\infty}$. Applying the theorem produces a subsequence $\{\vec{\mathbf{v}}_{N_p}\}_{p=1}^{\infty}$ limiting to some $\vec{\mathbf{v}}$ in K.

18. Verify identity
$\vec{\mathbf{v}}_N - A\vec{\mathbf{v}}_N = \frac{1}{N}(\vec{\mathbf{v}}_0 - A^N \vec{\mathbf{v}}_0)$.

19. Explain why $A\vec{\mathbf{v}} = \lim_{p\to\infty} A\vec{\mathbf{v}}_{N_p}$. Then prove $\vec{\mathbf{v}} = A\vec{\mathbf{v}}$.

20. The claimed eigenpair $(1, \vec{\mathbf{v}})$ has been found, provided $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$. Explain why $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$.

Coupled Systems

Find the coefficient matrix A. Identify as coupled or uncoupled and explain why.

21. $x' = 2x + 3y$, $y' = x + y$

22. $x' = 3y$, $y' = x$

23. $x' = 3x$, $y' = 2y$

24. $x' = 3x$, $y' = 2y$, $z' = z$

Solving Uncoupled Systems

Solve for the general solution.

25. $x' = 3x$, $y' = 2y$

26. $x' = 3x$, $y' = 2y$, $z' = z$

Change of Coordinates

Given the change of coordinates $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, find the matrix B for the inverse change $\vec{\mathbf{x}} = B\vec{\mathbf{y}}$.

27. $\vec{\mathbf{y}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$

28. $\vec{\mathbf{y}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathbf{x}}$

Constructing Coupled Systems

Given the uncoupled system and change of coordinates $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$, find the coupled system.

29. $x_1' = 2x_1$, $x_2' = 3x_2$, $P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

30. $x_1' = x_1$, $x_2' = -x_2$, $P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

Uncoupling a System

Change the given coupled system into an uncoupled system using the eigenanalysis change of variables $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$.

31. $x_1' = 2x_1$, $x_2' = x_1 + x_2$, $x_3' = x_3$
Ans: $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $y_1' = 2y_1$, $y_2' = y_2$, $y_3' = y_3$

32. $x_1' = x_1 + x_2$, $x_2' = x_1 + x_2$, $x_3' = x_3$
Ans: $P = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y_1' = 0$, $y_2' = 2y_2$, $y_3' = y_3$

Solving Coupled Systems

Report the answers for $x(t), y(t)$.

33. $x' = -x - 2y$, $y' = -4x + y$

34. $x' = 8x - y$, $y' = -2x + 7y$

Eigenanalysis and Footballs

The exercises study the ellipsoid $17x^2 + 8y^2 - 12xy + 80z^2 = 80$.

35. Let $A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A \vec{\mathbf{W}} = 80$, where $\vec{\mathbf{W}}$ has components x, y, z.

36. Find the eigenpairs of
$A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$.

37. Verify the semi-axis lengths $1, 4, 2$.

38. Verify that the ellipsoid has semi-axis unit directions
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

The Ellipse and Eigenanalysis

The exercises study the ellipse $2x^2 + 4xy + 5y^2 = 24$.

39. Let $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A \vec{\mathbf{W}} = 24$, where $\vec{\mathbf{W}} = \begin{pmatrix} x \\ y \end{pmatrix}$.

40. Find the eigenpairs of $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$.

41. Verify the semi-axis lengths $2, 2\sqrt{6}$.

42. Verify that the ellipse has semi-axis unit directions $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Orthogonal Triad Computation

The exercises fill in details from page 13, $x^2+4y^2+xy+4z^2=16$. Below,

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

43. Find the characteristic equation of A. Then verify the roots are 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

44. Show the steps from **rref** to second eigenvector $\vec{\mathbf{x}}_2$:

$$\mathbf{rref} = \begin{pmatrix} 1 & 3-\sqrt{10} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\vec{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{10}-3 \\ 1 \\ 0 \end{pmatrix}$$