# Chapter 9

# Eigenanalysis

**Contents**



This presentation of **matrix eigenanalysis** treats the subject in depth for a $3 \times 3$ matrix $A$. The generalization to an $n \times n$ matrix $A$ is to be supplied by the reader.

## 9.1 Eigenanalysis I

Studied here is eigenanalysis for matrix equations. The topics are *eigenanalysis*, *eigenvalue*, *eigenvector*, *eigenpair* and *diagonalization*.

### What's Eigenanalysis?

The term **eigenanalysis** refers to the identification and computation of a *new coordinate system*, along with *paired scale factors*, one scale factor per coordinate direction. The new coordinate system along with the units defined by the scale factors is employed to **simplify** the expression of the original mathematical model, be it a matrix model, a differential equations model, or otherwise.

**Matrix eigenanalysis** applies to a matrix equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$.

**Differential equation eigenanalysis** applies to an ordinary or partial differential equation.

### Matrix Eigenanalysis

Consider the matrix equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, where symbol $A$ is a square matrix of constants and symbols $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ are column vectors. The matrix equation is equivalent to simultaneous linear algebraic equations. For $3 \times 3$ matrices, the equivalent set of linear algebraic equations is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3. \end{cases}$$

**Table 1. What is Matrix Eigenanalysis?**

Matrix eigenanalysis identifies a change of variables that simplifies the linear simultaneous algebraic equations represented by the system $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$.

Eigenanalysis on a $3 \times 3$ matrix $A$ discovers a new coordinate system $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ which replaces the matrix equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ with simplified equations.

**Table 2. Simplification of $3 \times 3$ Linear Algebraic Equations**

A change of variables $\vec{\mathbf{X}} = P\vec{\mathbf{x}}$, $\vec{\mathbf{B}} = P\vec{\mathbf{b}}$. using eigenanalysis vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ for the columns of $P$, simplifies a $3\times 3$ system of linear algebraic equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ into the diagonal form

$$\begin{cases} \lambda_1 X_1 &= B_1, \\ \lambda_2 X_2 &= B_2, \\ \lambda_3 X_3 &= B_3. \end{cases} \tag{1}$$

Scalar values $\lambda_1, \lambda_2, \lambda_3$ are scale factors (measurement units) corresponding to the directions $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Precise definitions and details appear later.

## Coordinate Change via Eigenanalysis

Technically, matrix eigenanalysis is an opportunistic change of coordinates, which means the analysis must compute a set of independent column vectors that span the space. Linear algebra calls such a set of vectors a **basis**. Eigenanalysis constructs from the matrix $A$ a *special basis*. This basis defines a change of coordinates $\vec{\mathbf{x}} \to P\vec{\mathbf{x}}$ where $P$ is the augmented matrix of basis vectors.

Consider vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ which form a basis for $\mathcal{R}^3$. To be a *basis* means that *each possible* vector $\vec{\mathbf{x}}$ in $\mathcal{R}^3$ can be uniquely expressed as a linear combination $\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$. Geometrically, the vectors are a basis provided each possible $\vec{\mathbf{x}}$ in $\mathcal{R}^3$ can be constructed from the triad $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ using solely the geometric parallelogram law for vector addition.

The change of coordinates is defined by $P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle$ = augmented matrix of the three basis vectors. The claimed simplifying change of variables is $\vec{\mathbf{X}} = P\vec{\mathbf{x}}$, $\vec{\mathbf{Y}} = P\vec{\mathbf{y}}$, which changes $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ into $\vec{\mathbf{Y}} = B\vec{\mathbf{X}}$ where $B = PAP^{-1}$ (more details later).

## Fourier's Replacement Process

J. B. Fourier (1768-1830) discovered this process in his 1822 study of heat conduction. For the case of $\mathcal{R}^3$, basis vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are re-scaled by invented **scale factors** $\lambda_1, \lambda_2, \lambda_3$, which we imagine as measurement units along the three directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$. Briefly, a vector $\vec{\mathbf{x}}$ is replaced by a new vector $\vec{\mathbf{y}}$, according to the rule

$$\begin{array}{lcl} \vec{\mathbf{x}} & = & c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 \text{ implies} \\ \vec{\mathbf{y}} & = & c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3. \end{array} \tag{2}$$

**Table 3. Fourier's Re-Scaling Idea**

**Replace** $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ by re-scaled vectors $\lambda_1\vec{\mathbf{v}}_1$, $\lambda_2\vec{\mathbf{v}}_2$, $\lambda_3\vec{\mathbf{v}}_3$.

Critical readers can complain that no connection has been made between the matrix $A$, the basis vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the scale factors $\lambda_1, \lambda_2, \lambda_3$. For the moment, please focus on the *replacement process*, postponing the connection with $A$ until much later, when details will be revealed.

## A Key Example: Data Conversion

Let $\vec{\mathbf{x}}$ in $\mathcal{R}^3$ be a data set variable with coordinates $x_1$, $x_2$, $x_3$ recorded respectively in units of meters, millimeters and centimeters. Imagine the data being recorded every few milliseconds from three different sensors.

The $\vec{\mathbf{x}}$-data set is converted into a $\vec{\mathbf{y}}$-data set with meter, kilogram, second units (MKS units) via the equations

$$\left\{ \begin{array}{lcl} y_1 & = & x_1, \\ y_2 & = & 0.001x_2, \\ y_3 & = & 0.01x_3. \end{array} \right. \tag{3}$$

Equations (3) are an instance of Fourier's replacement process, Table 3. The paired scale factors and vectors are

$$\begin{array}{lll} \lambda_1 = 1, & \lambda_2 = 0.001, & \lambda_3 = 0.01, \\ \vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{array}$$

Then equations (3) can be written as the replacement process

$$
(4)\quad \begin{aligned}
\vec{\mathbf{x}} &= x_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + x_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + x_3 \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{implies}\\
\vec{\mathbf{y}} &= x_1\lambda_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\lambda_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\lambda_3 \begin{pmatrix}0\\0\\1\end{pmatrix}.
\end{aligned}
$$

Vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are the *data directions* (or *axes*) re-scaled by the measurement units $\lambda_1$, $\lambda_2$, $\lambda_3$, respectively. In particular, data direction $\vec{\mathbf{v}}_2$ is for millimeters.

## Fourier's Replacement Process as a Matrix Equation

**Theorem 1 (Matrix Form of Fourier's Replacement)**
Assume vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent. Fourier's replacement equation

$$
\begin{aligned}
\vec{\mathbf{x}} &= c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 \text{ implies}\\
\vec{\mathbf{y}} &= c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3
\end{aligned}
$$

has the matrix equivalent

$$
\vec{\mathbf{y}} = PDP^{-1}\vec{\mathbf{x}}, \quad \text{where} \quad P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle, \quad D = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}.
$$

**Proof**: The formula is justified from the scalar replacement equations by writing

$$
\vec{\mathbf{c}} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}, \quad \vec{\mathbf{y}} = P\begin{pmatrix}\lambda_1 c_1\\ \lambda_1 c_1\\ \lambda_1 c_1\end{pmatrix} = PD\vec{\mathbf{c}}.
$$

Then $\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = P\vec{\mathbf{c}}$ implies $\vec{\mathbf{c}} = P^{-1}\vec{\mathbf{x}}$, giving $\vec{\mathbf{y}} = PD\vec{\mathbf{c}} = PDP^{-1}\vec{\mathbf{x}}$, proving the formula in Theorem 1.

The matrix $P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle$ is the familiar augmented matrix of column vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. The matrix product $P\vec{\mathbf{c}}$ expands to $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$ because matrix multiply is a linear combination of the columns. The inverse $P^{-1}$ exists due to assumed independence of the three vectors.

## Definitions: Eigenvalue, Eigenvector and Eigenpair

**Eigenanalysis** for the matrix equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ is the algebraic method for discovering the basis vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the scale factors $\lambda_1, \lambda_2, \lambda_3$. The vectors are called **eigenvectors** and the scale factors are called **eigenvalues**.

A scale factor $\lambda$ is thought to be a **measurement unit** along an axis $\vec{\mathbf{v}}$, therefore the eigenvectors and eigenvalues occur in pairs, called **eigenpairs**. The dependence of a pair is due to the fundamental equation below, which is used in references to define and/or compute an eigenpair.

**Definition 1**
An **eigenpair** $(\lambda, \vec{\mathbf{v}})$ is defined to be a solution of the problem

$$A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}, \quad \vec{\mathbf{v}} \neq \vec{\mathbf{0}}. \tag{5}$$

Vector $\vec{\mathbf{v}}$ is called an **eigenvector**. The value $\lambda$ is called the **eigenvalue** corresponding to the eigenvector $\vec{\mathbf{v}}$.

The motivation for the rather abstract definition of eigenpair appears below. Excuses aside, the reader is asked to know definition (5), because of its explicit use in computations.

Eigenpair computation will be delayed, in order to explain the relevance in the $3\times 3$ case of equation (5) to the problem of computing basis vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and scale factors $\lambda_1, \lambda_2, \lambda_3$.

**Important**. Because $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ in equation (5), then an eigenvector is never the zero vector: an eigenvector is a *direction.* Otherwise stated, an eigenvector answer of zero means you just discovered an algebra error.

**History**. The German term *eigenwert* was coined by David Hilbert in 1904. James J. Sylvester in 1883 coined the equivalent term **latent root**:

> ...the latent roots of a matrix – latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf.

By 1967, Paul Halmos gave up the battle over which words to use in his text *A Hilbert Space Problem Book*:

> For many years I have battled for **proper values** (Sylvester's latent roots), and against the one and a half times translated German-English hybrid (eigenvalue) that is often used to refer to them. I have now become convinced that the war is over, and eigenvalues have won it; in this book I use them.

## Eigenpair Equations and $AP = PD$

The matrix formulation $\vec{\mathbf{y}} = PDP^{-1}\vec{\mathbf{x}}$ reduces to vector-matrix eigenpair equations, as follows. Suppose that $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$. Then $A\vec{\mathbf{x}} = PDP^{-1}\vec{\mathbf{x}}$ holds for all $\vec{\mathbf{x}}$, which in turn implies that $A = PDP^{-1}$, or equivalently,

$$AP = PD.$$

Using

$$P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

write the two matrix multiply equations $AP$ and $PD$ in expanded form

$$AP = \langle A\vec{\mathbf{v}}_1|A\vec{\mathbf{v}}_2|A\vec{\mathbf{v}}_3\rangle, \quad PD = \langle \lambda_1\vec{\mathbf{v}}_1|\lambda_2\vec{\mathbf{v}}_2|\lambda_3\vec{\mathbf{v}}_3\rangle.$$

Matching columns left and right in the equation $AP = PD$ then implies

$$A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1, \quad A\vec{\mathbf{v}}_2 = \lambda_2\vec{\mathbf{v}}_2, \quad A\vec{\mathbf{v}}_3 = \lambda_3\vec{\mathbf{v}}_3.$$

Reversibility of steps implies:

**Theorem 2 (Eigenpairs and $AP = PD$)**
Assume $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ independent. Then relations

$$\text{(6)} \qquad \left\{\begin{array}{rcl} A\vec{\mathbf{v}}_1 & = & \lambda_1\vec{\mathbf{v}}_1, \\ A\vec{\mathbf{v}}_2 & = & \lambda_2\vec{\mathbf{v}}_2, \\ A\vec{\mathbf{v}}_3 & = & \lambda_3\vec{\mathbf{v}}_3. \end{array}\right. \quad \textbf{(Eigenpair Equations)}$$

hold if and only if $AP = PD$ where $P$ and $D$ are defined by equations

$$\text{(7)} \qquad P = \langle \vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2|\vec{\mathbf{v}}_3\rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

## Computing Eigenpairs of a Matrix

To compute an eigenpair $(\lambda, \vec{\mathbf{v}})$ of a square matrix $A$ requires finding scalar $\lambda$ and a nonzero vector $\vec{\mathbf{v}}$ satisfying the homogeneous matrix–vector equation

$$A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}.$$

Write it as $A\vec{\mathbf{x}} - \lambda\vec{\mathbf{x}} = \vec{\mathbf{0}}$, then replace $\lambda\vec{\mathbf{x}}$ by $\lambda I\vec{\mathbf{x}}$ to obtain the standard linear algebraic system form[1]

$$(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{v}} \neq \vec{\mathbf{0}}.$$

Let $B = A - \lambda I$. The homogeneous equation $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has a nonzero solution $\vec{\mathbf{v}}$ if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occur if and only if $\mathbf{rref}(B)$ has a row of zeros. Determinant theory gives a more concise statement: $\det(B) = 0$ if and only if $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has infinitely many solutions. This proves:

**Theorem 3 (Characteristic Equation)**
The eigenvalues of a square matrix $A$ are roots $\lambda$ of the polynomial equation

$$\det(A - \lambda I) = 0.$$

[1] Identity $I$ is required to factor out the matrix $A - \lambda I$. It is wrong to factor out $A - \lambda$, because $A$ is $3 \times 3$ and $\lambda$ is $1 \times 1$, incompatible sizes for matrix addition.

The equation is called the **characteristic equation**. The **characteristic polynomial** is the polynomial obtained by determinant evaluation on the left, using cofactor expansion or the triangular rule.

**Characteristic Equation Illustration**.

$$\begin{aligned} \det\left(\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= \begin{vmatrix} 1-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda)-6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda+1)(\lambda-4). \end{aligned}$$

The characteristic equation $\lambda^2 - 3\lambda - 4 = 0$ has roots $\lambda_1 = -1$, $\lambda_2 = 4$. The characteristic polynomial is $\lambda^2 - 3\lambda - 4$.

**Table 4. Shortcut for the Characteristic Polynomial**

To find the characteristic polynomial $|A - \lambda I|$, subtract $\lambda$ from the diagonal of $A$ and then evaluate the determinant.

**Theorem 4 (Finding Eigenvectors of Matrix $A$)**
For each root $\lambda$ of the characteristic equation, write the toolkit sequence for $B = A - \lambda I$ ending with $\mathbf{rref}(B)$, followed by solving for the general solution $\vec{\mathbf{v}}$ of the homogeneous equation $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$, which contains invented symbols $t_1$, $t_2$, .... The vector basis answers $\partial_{t_1}\vec{\mathbf{v}}$, $\partial_{t_2}\vec{\mathbf{v}}$, ... are exactly all independent **eigenvectors** of $A$ paired to eigenvalue $\lambda$.[2]

**Proof**: The equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ is equivalent to $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Because $\det(B) = 0$, then this system has infinitely many solution. Then a toolkit sequence starting with $B$ and ending with $\mathbf{rref}(B)$ will have at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols $t_1, t_2, \ldots$, and then the vector basis is obtained by from the corresponding list of partial derivatives on these symbols. Each basis element is a nonzero solution of $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. By construction, the basis elements (eigenvectors for $\lambda$) are collectively independent. The proof is complete.

**Computational Complexity**. The theorem implies that a $3\times 3$ matrix $A$ with eigenvalues 1, 2, 3 causes three toolkit sequences to be computed, each sequence producing one eigenpair. In contrast, if $A$ has eigenvalues 1, 1, 1, then only one toolkit sequence is computed, resulting in one, two or three eigenpairs for eigenvalue $\lambda = 1$.

[2]Gilbert Strang has called these vector answers **special solutions** of the homogeneous equation $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Readers should justify that Strang's special solutions are independent and their linear combinations are all possible solutions of $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

## Fourier's Replacement for Matrices

Eigenanalysis has humble beginnings in **Fourier's Replacement Process**:

$$\begin{aligned}\vec{\mathbf{x}} &= c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 \text{ implies} \\ \vec{\mathbf{y}} &= c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3.\end{aligned} \tag{8}$$

If $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, where $A$ is a $3\times 3$ matrix, then these relations can be written as a single equation

$$A\left(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3\right) = c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3. \tag{9}$$

**Table 5. Fourier's Replacement for a Matrix $A$**

| |
|---|
| To compute $A\vec{\mathbf{x}}$ from $\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$, replace each vector $\vec{\mathbf{v}}_i$ by its scaled version $\lambda_i\vec{\mathbf{v}}_i$. |

The replacement process depends upon the three vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ being a basis for $\mathcal{R}^3$, in order that $\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1+c_2\vec{\mathbf{v}}_2+c_3\vec{\mathbf{v}}_3$ represents all possible vectors in $\mathcal{R}^3$.

For example, the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ fails to satisfy the replacement equation (9) for any choice of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ and scalars $\lambda_1, \lambda_2, \lambda_3$. The failure is exactly due to the lack of a full set of three eigenpairs: this example has just two eigenpairs.

### Fourier's Replacement Illustrated

Let

$$\begin{aligned} &A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix} \\ &\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = -5, \\ &\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}. \end{aligned} \tag{10}$$

Then Fourier's model holds (details delayed) and

$$\vec{\mathbf{x}} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

implies

$$A\vec{\mathbf{x}} = c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

## History of Fourier's Replacement

The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$.

> The vector $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ is obtained from eigenvalues $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_n$ and eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\ldots$, $\vec{\mathbf{v}}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\vec{\mathbf{v}}_1$, $\lambda_2\vec{\mathbf{v}}_2$, $\ldots$, $\lambda_n\vec{\mathbf{v}}_n$:
>
> $$\begin{array}{rcl} \vec{\mathbf{x}} &=& c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_n\vec{\mathbf{v}}_n \quad \text{implies} \\ \vec{\mathbf{y}} &=& x_1\lambda_1\vec{\mathbf{v}}_1 + x_2\lambda_2\vec{\mathbf{v}}_2 + \cdots + c_n\lambda_n\vec{\mathbf{v}}_n. \end{array}$$

## Powers and Fourier's Replacement

Powers $A^n$ of a matrix $A$ for which Fourier's Replacement equation (9) holds can be computed using only the basic vector space toolkit. To illustrate, assume $3 \times 3$ matrix $A$ has eigenpairs $(\lambda_1, \vec{\mathbf{v}}_i)$, $i = 1, 2, 3$. Then only the vector toolkit for $\mathcal{R}^3$ is used in computing

$$A^5\vec{\mathbf{x}} = x_1\lambda_1^5\vec{\mathbf{v}}_1 + x_2\lambda_2^5\vec{\mathbf{v}}_2 + x_3\lambda_3^5\vec{\mathbf{v}}_3.$$

This calculation does not use matrix multiply and it does not depend upon finding previous powers $A^2$, $A^3$, $A^4$.

Fourier's replacement reduces computational effort. Matrix–vector multiplication to produce $\vec{\mathbf{y}}_k = A^k\vec{\mathbf{x}}$ requires $9k$ multiply operations whereas Fourier replacement gives the answer with $3k + 9$ multiply operations.

## Differential Equations and Fourier's Replacement

Systems of differential equations can be solved using Fourier's replacement, giving a compact and elegant formula for the general solution. An example:

$$\begin{array}{rcl} x_1' &=& x_1 + 3x_2, \\ x_2' &=& 2x_2 - x_3, \\ x_3' &=& -5x_3. \end{array}$$

The matrix form is $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$, where $A$ and its eigenpairs are defined in equation (10).

Fourier's re-scaling idea applies to linear differential equations, as follows. First, expand the initial condition $\vec{\mathbf{x}}(0)$ in terms of basis elements $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$:

$$\vec{\mathbf{x}}(0) = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3.$$

Then the general solution of $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$ is given by replacing each $\vec{\mathbf{v}}_i$ by the re-scaled vector $e^{\lambda_i t}\vec{\mathbf{v}}_i$, resulting in the formula

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3.$$

For the illustration here, the result is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t}\begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

## Independence of Eigenvectors

**Theorem 5 (Independence of Eigenvectors)**
If $(\lambda_1, \vec{\mathbf{v}}_1)$ and $(\lambda_2, \vec{\mathbf{v}}_2)$ are two eigenpairs of $A$ and $\lambda_1 \ne \lambda_2$, then $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are linearly independent vectors.

More generally, if $(\lambda_1, \vec{\mathbf{v}}_1)$, ..., $(\lambda_k, \vec{\mathbf{v}}_k)$ are eigenpairs of $A$ corresponding to distinct eigenvalues $\lambda_!$, ..., $\lambda_k$, then $\vec{\mathbf{v}}_1$, ..., $\vec{\mathbf{v}}_k$ are independent.

**Proof**: Let's solve $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$ for $c_1$, $c_2$. The vectors are independent provided the only solution is $c_1 = c_2 = 0$. Apply $A$ to this equation, obtaining $c_1 A\vec{\mathbf{v}}_1 + c_2 A\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$. Use $A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1$ and $A\vec{\mathbf{v}}_2 = \lambda_2\vec{\mathbf{v}}_2$ to obtain $c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$. Multiply $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$ by $\lambda_1$ and subtract it from $c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$ to get $c_1(\lambda_1 - \lambda_1)\vec{\mathbf{v}}_1 + c_2(\lambda_2 - \lambda_1)\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$. Because $\lambda_2 \ne \lambda_1$, cancel $\lambda_2 - \lambda_1$ to give $c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$. The assumption $\vec{\mathbf{v}}_2 \ne \vec{\mathbf{0}}$ implies $c_2 = 0$. Return to the first equation and use $c_2 = 0$ to obtain $c_1\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. Because $\vec{\mathbf{v}}_1 \ne \vec{\mathbf{0}}$, then $c_1 = 0$. This proves $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent.

The general case is proved by induction on $k$. The case $k = 1$ follows because a nonzero vector is an independent set. Assume it holds for $k - 1$ and let's prove it for $k$, when $k > 1$. To prove independence, solve

$$c_1\vec{\mathbf{v}}_1 + \cdots + c_k\vec{\mathbf{v}}_k = \vec{\mathbf{0}}$$

for $c_1$, ..., $c_k$. Create a second equation by multiplication by $A$, effectively replacing each $c_i$ by $\lambda_i c_i$ (because $A\vec{\mathbf{v}}_i = \lambda_i\vec{\mathbf{v}}_i$). Then multiply the first equation by $\lambda_1$ and subtract the two equations to get

$$c_1(\lambda_1 - \lambda_1)\vec{\mathbf{v}}_1 + c_2(\lambda_1 - \lambda_2)\vec{\mathbf{v}}_2 + \cdots + c_k(\lambda_1 - \lambda_k)\vec{\mathbf{v}}_k = \vec{\mathbf{0}}.$$

The first term is zero. Apply the induction hypothesis to the remaining $k - 1$ vectors, then all coefficients $(\lambda_1 - \lambda_i)c_i$ are zero. Because $\lambda_1 - \lambda_i \ne 0$ for $i > 1$, then $c_2$ through $c_k$ are zero. Return to the first equation to obtain $c_1\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. Because $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$, then $c_1 = 0$. The induction is complete.

**Table 6. How to Find a Complete Set of Independent Eigenvectors**

Solve the characteristic equation $|A - \lambda I| = 0$ for all eigenvalues $\lambda$. For each $\lambda$, let $B = A - \lambda I$ and solve $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$ for general solution $\vec{\mathbf{v}}$, which contains invented symbols $t_1, t_2, \ldots$. Append the partial derivatives $\partial_{t_1}\vec{\mathbf{v}}, \partial_{t_2}\vec{\mathbf{v}}, \ldots$ to the set of all eigenvectors for eigenvalue $\lambda$. Then the complete set of eigenvectors so obtained is independent.

### Eigenanalysis Facts

1. An **eigenvalue** $\lambda$ of a triangular matrix $A$ is one of the diagonal entries. If $A$ is non-triangular, then an eigenvalue is found as a root $\lambda$ of the characteristic equation $|A - \lambda I| = 0$.

2. An **eigenvalue** of a square matrix $A$ can be zero, positive, negative or even complex. It is a pure number, with a physical meaning inherited from the model, e.g., a scale factor or measurement unit.

3. An **eigenvector** for eigenvalue $\lambda$ (a scale factor) is a nonzero direction $\vec{\mathbf{v}}$ of application satisfying $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. It is found from a toolkit sequence starting at $B = A - \lambda I$ and ending at $\mathbf{rref}(B)$. Independent eigenvectors are computed from the general solution as partial derivatives $\partial/\partial t_1$, $\partial/\partial t_2$, ....

4. If a $3 \times 3$ matrix has three independent eigenvectors, then they collectively form a **basis** of $\mathcal{R}^3$ (a **coordinate system**).

## Eigenanalysis and Geometry

In case the matrix $A$ is $2 \times 2$ or $3 \times 3$, geometry can provide additional intuition about eigenanalysis.

Fourier's $2 \times 2$ replacement $A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$ can be interpreted as the action of the transformation $T : \vec{x} \to A\vec{x}$ between two copies of the plane $\mathcal{R}^2$; see Figure 1.

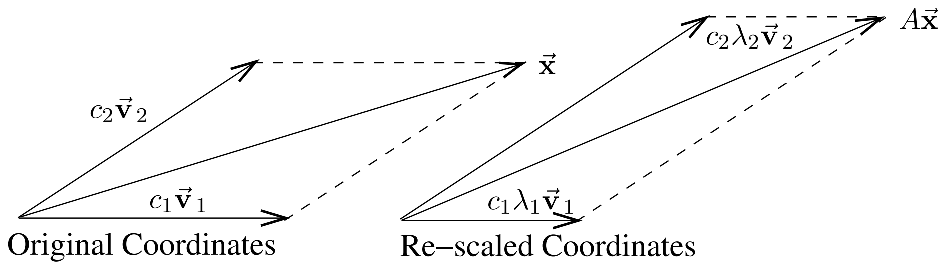


**Figure 1. Transformation** $T:\ \mathcal{R}^2 \to \mathcal{R}^2$**.**
Vector $\vec{\mathbf{x}}$ is obtained geometrically from $\vec{\mathbf{v}}_1.\vec{\mathbf{v}}_2$ by changing their lengths by $c_1, c_2$, then add with the parallelogram rule. Vector $A\vec{\mathbf{x}}$ is obtained from the two changed vectors by re-scaling by $\lambda_1, \lambda_2$, then apply the parallelogram rule.

Algebraically, $A$ is replaced by the scale factors $\lambda_1, \lambda_2$ and the coordinate system $\vec{v}_1, \vec{v}_2$. The **eigenvalues** are the scale factors. The two vectors used in the parallelogram rule are the **eigenvectors**.

## Shears are not Fourier Replacement

The important geometrical operations are scaling, shears, rotations, projections, reflections and translations. Geometrically talented readers will recognize that Fourier's replacement process describes scaling along coordinate axes.

A planar horizontal shear $(x_1, x_2) \to (y_1, y_2)$ is a set of equations

$$\begin{aligned} y_1 &= x_1 + kx_2, \quad (k = \text{shear factor} \neq 0), \\ y_2 &= x_2. \end{aligned}$$

The eigenvalues of $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ are $\lambda_1 = \lambda_2 = 1$. If it is possible to view a shear as a re-scaling, then it must be possible to change coordinates to new independent axes $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ and express the shear as

$$A = PDP^{-1}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \rangle.$$

Then $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = A = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction to $k \neq 0$.

Conclusion: A shear is not equivalent to scaling along axes: Fourier's replacement fails.

## Diagonalization and Eigenpair Packages

All results in this subsection are valid for $n \times n$ matrices. The results are sometimes stated for the $3 \times 3$ case, to improve clarity.

**Definition 2 (Diagonalizable Matrix)**
An $n \times n$ matrix $A$ which has $n$ independent eigenvectors is called **diagonalizable**.

In practical terms, the augmented matrix $P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle$ of eigenvectors provides a variable change $\vec{\mathbf{X}} = P\vec{\mathbf{x}}, \vec{\mathbf{Y}} = P\vec{\mathbf{y}}$ to transform the $3 \times 3$ system $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ into $\vec{\mathbf{Y}} = D\vec{\mathbf{X}}$, where is a diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

**Theorem 6 (Diagonalization and Diagonal Matrices)**
A $3 \times 3$ diagonal matrix $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ has eigenvalues on the diagonal and the corresponding eigenvectors are the columns of the $3 \times 3$ identity matrix. In summary, the eigenpairs of $A$ are

$$\begin{array}{lll} \lambda_1 = a, & \lambda_2 = b, & \lambda_3 = c, \\ \vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{array}$$

**Definition 3 (Eigenpair Packages)**
Let $A$ be a diagonalizable $3 \times 3$ matrix with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$, $(\lambda_3, \vec{\mathbf{v}}_3)$. Define the **eigenpair packages**[3]

$$P = \langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

**Theorem 7 (Diagonalization)**
Let $A$ be a diagonalizable $n \times n$ matrix with eigenpair packages $P$, $D$.

1. The matrix $A$ is completely determined by its eigenpairs:

$$A = PDP^{-1}.$$

2. The change of variables $\vec{\mathbf{X}} = Px$, $\vec{\mathbf{Y}} = P\vec{\mathbf{y}}$ transforms the equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ into the diagonal system $\vec{\mathbf{Y}} = D\vec{\mathbf{X}}$.

[3]Eigenpair packages are not unique. For $3 \times 3$, there are six (6) permutations of the pairs, leading to six different packages. In addition, eigenvectors are not unique, leading to infinity many possible eigenpair packages.

**3**. The equation $A(c_1\vec{\mathbf{v}}_1 + \cdots + c_n\vec{\mathbf{v}}_n) = c_1\lambda_1\vec{\mathbf{v}}_1 + \cdots + c_n\lambda_n\vec{\mathbf{v}}_n$ of Fourier replacement holds with matrix form

$$AP\vec{\mathbf{c}} = PD\vec{\mathbf{c}}, \quad \vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \text{arbitrary vector in } \mathcal{R}^n.$$

**Theorem 8 (Distinct Eigenvalues)**
If an $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then it has $n$ eigenpairs $(\lambda_i, \vec{\mathbf{v}}_i)$, $i = 1, \ldots, n$. The eigenpair packages

$$P = \langle \vec{\mathbf{v}}_1 | \cdots | \vec{\mathbf{v}}_n \rangle, \quad D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

satisfy $AP = PD$ and matrix $A$ is diagonalizable.

## Examples and Computational Details

**1 Example (Computing $2 \times 2$ Eigenpairs)**
Find all eigenpairs of the $2 \times 2$ matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

**Solution**:
**College Algebra**. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Details:

| | |
|---|---|
| $0 = \det(A - \lambda I)$ | Characteristic equation. |
| $= \begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix}$ | Subtract $\lambda$ from the diagonal. |
| $= (1-\lambda)(-1-\lambda)$ | Sarrus' rule. |

**Linear Algebra**. The eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Details:
**Eigenvector for** $\lambda_1 = 1$.

| | |
|---|---|
| $A - \lambda_1 I = \begin{pmatrix} 1-\lambda_1 & 0 \\ 2 & -1-\lambda_1 \end{pmatrix}$ | |
| $= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}$ | |
| $\approx \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ | Swap and multiply rules. |
| $= \mathbf{rref}(A - \lambda_1 I)$ | Reduced echelon form. |

The vector partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the scalar general solution $x = t_1$, $y = t_1$ is eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

**Eigenvector for** $\lambda_2 = -1$.

$$\begin{aligned} A - \lambda_2 I &= \begin{pmatrix} 1-\lambda_2 & 0 \\ 2 & -1-\lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\ &= \mathbf{rref}(A - \lambda_2 I) && \text{Reduced echelon form.} \end{aligned}$$

The vector partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the scalar general solution $x = 0$, $y = t_1$ is eigenvector $\vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

**2 Example (Computing $2 \times 2$ Complex Eigenpairs)**

Find all eigenpairs of the $2 \times 2$ matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

**Solution**:

**College Algebra**. The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= (1-\lambda)^2 + 4 && \text{Sarrus' rule.} \end{aligned}$$

The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2 + 4 = 0$. Alternatively, use $(1-\lambda)^2 = -4$ and take square roots.

**Linear Algebra**. The eigenpairs are $\left(1 + 2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$, $\left(1 - 2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$.

**Eigenvector for** $\lambda_1 = 1 + 2i$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 2 \\ -2 & 1-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \\ &\approx \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} && \text{Multiply rule.} \\ &\approx \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix} && \text{Combination rule, multiplier=}-i. \\ &\approx \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} && \text{Swap rule.} \\ &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the general solution $x = -it_1$, $y = t_1$ is eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

**Eigenvector for** $\lambda_2 = 1 - 2i$.
The problem $(A - \lambda_2 I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has solution $\vec{\mathbf{v}} = \overline{\vec{\mathbf{v}}_1}$. To see why, takes conjugates[4] across the equation to get $(A - \overline{\lambda_2} I)\overline{\vec{\mathbf{v}}} = \vec{\mathbf{0}}$. Then $\lambda_1 = \overline{\lambda_2}$ gives $\vec{\mathbf{v}} = \overline{\vec{\mathbf{v}}_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

### 3 Example (Computing $3 \times 3$ Eigenpairs)

Find all eigenpairs of the $3 \times 3$ matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

**Solution**:
**College Algebra**. The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= ((1-\lambda)^2 + 4)(3 - \lambda) && \text{Cofactor rule and Sarrus' rule.} \end{aligned}$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2 + 4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

**Linear Algebra**.

The eigenpairs are $\left(1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.

**Eigenvector for** $\lambda_1 = 1 + 2i$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 2 & 0 \\ -2 & 1-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{pmatrix} && \text{Subtract } \lambda_1 = 1 + 2i \text{ from the diagonal.} \\ &\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule.} \\ &\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Combination rule, factor=}-i\text{.} \\ &\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \end{aligned}$$

[4]Overbar $\overline{z}$ is used to indicate the complex conjugate of $z$, which for vectors is formally obtained componentwise by replacing complex unit $i$ by $-i$ (reflection through the $x$-axis). For instance, $\overline{2 - 3i}$ equals $2 + 3i$ (replace $i$ by $-i$).

$$= \mathbf{rref}(A - \lambda_1 I) \qquad \text{Reduced echelon form.}$$

The vector partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the scalar general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

**Eigenvector for** $\lambda_2 = 1 - 2i$.

There is one eigenpair $(1 - 2i, \vec{\mathbf{v}}_2)$, where $\vec{\mathbf{v}}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

**Details**. To see why, take conjugates[5] across the equation $(A - \lambda_2 I)\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$ to give $(\overline{A} - \overline{\lambda_2} I)\overline{\vec{\mathbf{v}}_2} = \vec{\mathbf{0}}$. Then $\overline{A} = A$ ($A$ is real) and $\lambda_1 = \overline{\lambda_2}$ gives $(A - \lambda_1 I)\overline{\vec{\mathbf{v}}_2} = \vec{\mathbf{0}}$. Then $\overline{\vec{\mathbf{v}}_2} = \vec{\mathbf{v}}_1$. Finally, $\vec{\mathbf{v}}_2 = \overline{\overline{\vec{\mathbf{v}}_2}} = \overline{\vec{\mathbf{v}}_1} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

**Eigenvector for** $\lambda_3 = 3$.

$$\begin{aligned}
A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 \\ -2 & 1-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\
&= \mathbf{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
\end{aligned}$$

The partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector $\vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

**4 Example (Data Conversion)** The data conversion problem

$$\begin{cases} y_1 &= x_1, \\ y_2 &= 0.001x_2, \\ y_3 &= 0.01x_3. \end{cases}$$

is diagonalizable. The three eigenpairs of $A$ are defined by

$$\begin{matrix} \lambda_1 = 1, & \lambda_2 = 0.001, & \lambda_3 = 0.01, \\ \vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{matrix}$$

[5]The complex conjugate is defined by $\overline{a + ib} = a - ib$. Conjugate rules apply to vectors componentwise. Two useful rules are $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

**Solution**: The example was introduced above in equation (3). These equations can be written as $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-2} \end{pmatrix}$ is already a diagonal matrix.

The reader is expected to apply supporting theorems to find the eigenpairs. For practise, the eigenpairs of $A$ can be found as in the previous example, an exercise which can discover the proof of the basic results about eigenpairs of diagonal matrices.

The answers should be verified directly from the eigenpair equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. For instance, when $\vec{\mathbf{v}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\lambda = 0.001$, then the two sides $A\vec{\mathbf{v}}$ and $\lambda\vec{\mathbf{v}}$ are computed from matrix multiply, each giving the same answer $\begin{pmatrix} 0 \\ 0.001 \\ 0 \end{pmatrix}$, therefore $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ is valid and $(\lambda, \vec{\mathbf{v}})$ is an eigenpair of $A$.

**5 Example (Decomposition $A = PDP^{-1}$)**

Decompose $A = PDP^{-1}$ where $P$, $D$ are eigenvector and eigenvalue packages, respectively, for the $3 \times 3$ matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Write explicitly Fourier's replacement equation in vector-matrix notation.

**Solution**: By the preceding example, the eigenpairs are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

The packages are therefore

$$D = \mathbf{diag}(1+2i, 1-2i, 3), \quad P = \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Fourier's replacement equation**. The action of $A$ in the model

$$A\left(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3\right) = c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3$$

is to replace the basis $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ by scaled vectors $\lambda_1\vec{\mathbf{v}}_1$, $\lambda_2\vec{\mathbf{v}}_2$, $\lambda_3\vec{\mathbf{v}}_3$. In vector form, the model is

$$AP\vec{\mathbf{c}} = PD\vec{\mathbf{c}}, \quad \vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then the action of $A$ is to replace eigenvector package $P$ by the re-scaled package $PD$. Explicitly,

$$\begin{aligned}
\vec{\mathbf{x}} &= c_1\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{implies} \\
A\vec{\mathbf{x}} &= c_1(1+2i)\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2(1-2i)\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3(3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

**6 Example (Diagonalization I)**
Report **diagonalizable** or **non-diagonalizable** for the $4 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If $A$ is diagonalizable, then assemble and report eigenvector and eigenvalue packages $P$, $D$.

**Solution**: The matrix $A$ is **non-diagonalizable**, because it fails to have 4 eigenpairs. The details:

**Eigenvalues**.

$$\begin{aligned}
0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\
&= \begin{vmatrix} 1-\lambda & 2 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} (3-\lambda)^2 && \text{Cofactor expansion applied twice.} \\
&= \left((1-\lambda)^2 + 4\right)(3-\lambda)^2 && \text{Sarrus' rule.}
\end{aligned}$$

The roots are $1 \pm 2i$, 3, 3, listed according to multiplicity.

**Eigenpairs**. They are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right).$$

Because only three eigenpairs exist, instead of four, then the matrix $A$ is **non-diagonalizable**. Details:

**Eigenvector for** $\lambda_1 = 1 + 2i$.

$$A - \lambda_1 I = \begin{pmatrix} 1-\lambda_1 & 2 & 0 & 0 \\ -2 & 1-\lambda_1 & 0 & 0 \\ 0 & 0 & 3-\lambda_1 & 1 \\ 0 & 0 & 0 & 3-\lambda_1 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & 2-2i & 1 \\ 0 & 0 & 0 & 2-2i \end{pmatrix} \\
&\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 2-2i & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule, three times.} \\
&\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
&\approx \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
&= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}
$$

The general solution is $x_1 = -it_1$, $x_2 = t_1$, $x_3 = 0$, $x_4 = 0$. Then $\partial_{t_1}$ applied to this solution gives the reported eigenpair for $\lambda = 1 + 2i$.

**Eigenvector for** $\lambda_2 = 1 - 2i$.
Because $\lambda_2$ is the conjugate of $\lambda_1$ and $A$ is real, then an eigenpair for $\lambda_2$ is found by taking the complex conjugate of the eigenpair reported for $\lambda_1$.

**Eigenvector for** $\lambda_3 = 3$. In theory, there can be one or two eigenpairs to report. It turns out there is only one, because of the following details.

$$
\begin{aligned}
A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 & 0 \\ -2 & 1-\lambda_3 & 0 & 0 \\ 0 & 0 & 3-\lambda_3 & 1 \\ 0 & 0 & 0 & 3-\lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule, two times.} \\
&\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply rule.} \\
&= \mathbf{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
\end{aligned}
$$

Apply $\partial_{t_1}$ to the general solution $x_1 = 0$, $x_2 = 0$, $x_3 = t_1$, $x_4 = 0$ to give the eigenvector matching the eigenpair reported above for $\lambda = 3$.

**7 Example (Diagonalization II)**
Report **diagonalizable** or **non-diagonalizable** for the $4 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If $A$ is diagonalizable, then assemble and report eigenvalue and eigenvector packages $D$, $P$.

**Solution**: The matrix $A$ is **diagonalizable**, because it has 4 eigenpairs

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Then the eigenpair packages are given by

$$D = \begin{pmatrix} -1+2i & 0 & 0 & 0 \\ 0 & 1-2i & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -i & i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The details parallel the previous example, except for the calculation of eigenvectors for $\lambda_3 = 3$. In this case, the reduced echelon form has two rows of zeros and parameters $t_1$, $t_2$ appear in the general solution. The answers given above for eigenvectors correspond to the partial derivatives $\partial_{t_1}$, $\partial_{t_2}$.

**8 Example (Non-diagonalizable Matrices)**
Verify that the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are all non-diagonalizable.

**Solution**: Let $A$ denote any one of these matrices and let $n$ be its dimension.

First, we will decide on diagonalization, without computing eigenpairs. Assume, in order to reach a contradiction, that eigenpair packages $D$, $P$ exist with $D$ diagonal and $P$ invertible such that $AP = PD$. Because $A$ is triangular, its eigenvalues appear already on the diagonal of $A$. Only 0 is an eigenvalue and its multiplicity is $n$. Then the package $D$ of eigenvalues is the zero matrix and an equation $AP = PD$ reduces to $AP = 0$. Multiply $AP = 0$ by $P^{-1}$ to obtain $A = 0$. But $A$ is not the zero matrix, a contradiction. We conclude that $A$ is not diagonalizable.

Second, we attack the diagonalization question directly, by solving for the eigenvectors corresponding to $\lambda = 0$. The toolkit sequence starts with $B = A - \lambda I$, but $B$ equals $\mathbf{rref}(B)$ and no computations are required. The resulting reduced echelon system is just $x_1 = 0$, giving $n-1$ free variables. Therefore, the eigenvectors of $A$ corresponding to $\lambda = 0$ are the last $n-1$ columns of the identity matrix $I$. Because $A$ does not have $n$ independent eigenvectors, then $A$ is not diagonalizable.

Similar examples of non-diagonalizable matrices $A$ can be constructed with $A$ having from 1 up to $n-1$ independent eigenvectors. The examples with ones on the super-diagonal and zeros elsewhere have exactly one eigenvector.

**9 Example (Fourier's 1822 Heat Model)**

Fourier's 1822 treatise *Théorie analytique de la chaleur* studied dissipation of heat from a laterally insulated welding rod with ends held at $0°$C (ice-packed ends). Assume the initial heat distribution along the rod at time $t = 0$ is given as a linear combination

$$f = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3.$$

Symbols $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are in the vector space $V$ of all twice continuously differentiable functions on $0 \le x \le 1$, given explicitly as

$$\vec{\mathbf{v}}_1 = \sin \pi x, \quad \vec{\mathbf{v}}_2 = \sin 2\pi x, \quad \vec{\mathbf{v}}_3 = \sin 3\pi x.$$

**Fourier's heat model** re-scales[6] each of these vectors to obtain the temperature $u(t,x)$ at position $x$ along the rod and time $t > 0$ as the model equation

$$u(t,x) = c_1 e^{-\pi^2 t}\vec{\mathbf{v}}_1 + c_2 e^{-4\pi^2 t}\vec{\mathbf{v}}_2 + c_3 e^{-9\pi^2 t}\vec{\mathbf{v}}_3.$$

Verify that $u(t,x)$ solves Fourier's partial differential equation heat model

$$\begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) &=& f(x), \quad 0 \le x \le 1, \quad \text{initial temperature}, \\ u(t,0) &=& 0, \quad \text{zero Celsius at rod's left end}, \\ u(t,1) &=& 0, \quad \text{zero Celsius at rod's right end}. \end{array}$$

**Solution**: First, we prove that the partial differential equation is satisfied by Fourier's solution $u(t,x)$. This is done by expanding the left side (LHS) and right side (RHS) of the differential equation, separately, then comparing the answers for equality.

Trigonometric functions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are solutions of three different linear ordinary differential equations: $u'' + \pi^2 u = 0$, $u'' + 4\pi^2 u = 0$, $u'' + 9\pi^2 u = 0$. Because of these differential equations, we can compute directly

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 c_1 e^{-\pi^2 t}\vec{\mathbf{v}}_1 - 4\pi^2 c_2 e^{-4\pi^2 t}\vec{\mathbf{v}}_2 - 9\pi^2 c_3 e^{-9\pi^2 t}\vec{\mathbf{v}}_3.$$

[6] The scale factors are not constants nor are they eigenvalues, but rather, they are exponential functions of $t$, as was the case for matrix differential equations $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Similarly, computing $\partial_t u(t,x)$ involves just the differentiation of exponential functions, giving

$$\frac{\partial u}{\partial t} = -\pi^2 c_1 e^{-\pi^2 t}\vec{\mathbf{v}}_1 - 4\pi^2 c_2 e^{-4\pi^2 t}\vec{\mathbf{v}}_2 - 9\pi^2 c_3 e^{-9\pi^2 t}\vec{\mathbf{v}}_3.$$

Because the second display is exactly the first, then LHS = RHS, proving that the partial differential equation is satisfied.

The relation $u(0,x) = f(x)$ is proved by observing that each exponential factor becomes $e^0 = 1$ when $t = 0$.

The two relations $u(t,0) = u(t,1) = 0$ hold because each of $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ vanish at $x = 0$ and $x = 1$. The verification is complete.

**The exercises are in progress, mostly incomplete. August 2016.**

## Exercises 9.1

**Eigenanalysis.** Classify as true or false. If false, then correct the text to make it true.

**1.** The purpose of eigenanalysis is to discover a new coordinate system.

**2.** Eigenanalysis can discover an opportunistic change of coordinates.

**3.** Diagonal matrices have all their eigenvalues on the last row.

**4.** Eigenvalues are scale factors, imagined to be measurement units.

**5.** For each eigenvalue of a matrix $A$, there always exists at least one eigenpair.

**6.** Eigenvectors are independent directions.

**7.** Eigenvalues of a square matrix cannot be zero.

**8.** Eigenvectors of a square matrix cannot be zero.

**9.** Eigenpairs $(\lambda, \vec{\mathbf{v}})$ of a square matrix $A$ satisfy the homogeneous equation $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

**Eigenpairs of a Diagonal Matrix.** Find the eigenpairs of $A$.

**10.** $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

**11.** $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

**12.** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

**13.** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

**14.** $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

**15.** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

**Fourier's Replacement Process.**

**16.**

**Eigenanalysis Facts.**

**17.**

**Eigenpair Packages.**

**18.**

**The Equation $AP = PD$.**

**19.**

**Matrix Eigenanalysis Method.**

**20.**

Basis of Eigenvectors.

**21.**

Independence of Eigenvectors.

**22.**

Diagonalization Theory.

**23.**

Non-diagonalizable Matrices.

**24.**

Distinct Eigenvalues.

**25.** $\begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$

**26.** $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

**27.** $\begin{pmatrix} 2 & 6 & 2 \\ 9 & 3 & 9 \\ 1 & 3 & 1 \end{pmatrix}$

**28.** $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

**29.** $\begin{pmatrix} 7 & 12 & 6 \\ 2 & 2 & 2 \\ -7 & -12 & -6 \end{pmatrix}$

**30.** $\begin{pmatrix} 2 & 2 & -6 \\ -3 & -4 & 3 \\ -3 & -4 & -1 \end{pmatrix}$

Computing $2 \times 2$ Eigenpairs.

**31.**

Computing $2 \times 2$ Complex Eigenpairs.

**32.**

Computing $3 \times 3$ Eigenpairs.

**33.** Suppose $A$ is row-reduced to a triangular form $B$. Are the eigenvalues of $B$ also the eigenvalues of $A$? Give a proof or a counter-example.

**34.**

**35.**

Decomposition $A = PDP^{-1}$.

**36.**

Diagonalization I

**37.**

Diagonalization II

**38.**

Non-diagonalizable Matrices

**39.** Invent a non-diagonalizable $3 \times 3$ matrix which has exactly one eigenpair.

Fourier's Heat Model

**40.**