Vector Spaces and Subspaces

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Vector Space V

It is a **data set** V plus a **toolkit** of eight (8) algebraic properties. The data set consists of packages of data items, called **vectors**, denoted $\vec{X}$, $\vec{Y}$ below.

Closure	The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$	commutative
	$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$	negative
Scalar multiply	$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$	distributive I
	$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X}) = (k_1 k_2)\vec{X}$	distributive III
	$1\vec{X} = \vec{X}$	identity

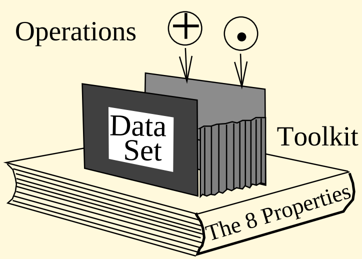

Figure 1. A *Vector Space* is a data set, operations $\boxplus$ and $\boxdot$, and the 8-property toolkit.

Definition of Subspace

A **subspace** $\boldsymbol{S}$ of a vector space $\boldsymbol{V}$ is a nonvoid subset of $\boldsymbol{V}$ which under the operations $+$ and $\cdot$ of $\boldsymbol{V}$ forms a vector space in its own right.

Subspace Criterion

Let $\boldsymbol{S}$ be a subset of $\boldsymbol{V}$ such that

1. Vector $\vec{\mathbf{0}}$ is in $\boldsymbol{S}$.
2. If $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$ are in $\boldsymbol{S}$, then $\vec{\boldsymbol{X}} + \vec{\boldsymbol{Y}}$ is in $\boldsymbol{S}$.
3. If $\vec{\boldsymbol{X}}$ is in $\boldsymbol{S}$, then $\boldsymbol{c}\vec{\boldsymbol{X}}$ is in $\boldsymbol{S}$.

Then $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$.

Items **2**, **3** can be summarized as *all linear combinations of vectors in* $\boldsymbol{S}$ *are again in* $\boldsymbol{S}$. In proofs using the criterion, items 2 and 3 may be replaced by

$$\boldsymbol{c}_1\vec{\boldsymbol{X}} + \boldsymbol{c}_2\vec{\boldsymbol{Y}} \text{ is in } \boldsymbol{S}.$$

Subspaces are Working Sets

We call a subspace $\boldsymbol{S}$ of a vector space $\boldsymbol{V}$ a **working set**, because the purpose of identifying a subspace is to shrink the original data set $\boldsymbol{V}$ into a smaller data set $\boldsymbol{S}$, customized for the application under study.

A Key Example. Let $\boldsymbol{V} = \boldsymbol{R}^3$ and let $\boldsymbol{S}$ be the plane of action of a planar kinematics experiment, a slot car on a track. The data set $\boldsymbol{D}$ for the experiment is all $\mathbf{3}$-vectors $\vec{\boldsymbol{v}}$ in $\boldsymbol{V}$ collected by a data recorder. Detected by GPS and recorded by computer is the $\mathbf{3}$D position of the slot car, which would be planar, except for bumps in the track. The original data set $\boldsymbol{D}$ will be transformed into a new data set $\boldsymbol{D}_1$ that lies entirely in a plane. Plane geometry computations then proceed with the Toolkit for $\boldsymbol{V}$, on the smaller planar data set $\boldsymbol{D}_1$. *How to create* $\boldsymbol{D}_1$? The ideal plane of action $\boldsymbol{S}$ is computed as a homogeneous equation (like 2x+3y+1000z=0), the equation of a plane. Least squares is applied to data set $\boldsymbol{D}$ to find an optimal equation for $\boldsymbol{S}$. Altered data set $\boldsymbol{D}_1$ is created from $\boldsymbol{D}$ (and $\boldsymbol{D}$ discarded) to artificially satisfy the plane equation. Then $\boldsymbol{D}_1$ is contained in $\boldsymbol{S}$ ($\boldsymbol{D}$ obviously was not). The smaller storage set $\boldsymbol{S}$ replaces the larger storage set $\boldsymbol{V}$. The customized smaller set $\boldsymbol{S}$ is the *working set* for the kinematics problem.

The Kernel Theorem

Theorem 1 (Kernel Theorem)
Let V be one of the vector spaces R^n and let A be an $m \times n$ matrix. Define a smaller set S of data items in V by the kernel equation

$$S = \{\vec{x} \,:\, \vec{x} \text{ in } V, \quad A\vec{x} = \vec{0}\}.$$

Then S is a subspace of V.
In particular, operations of addition and scalar multiplication applied to data items in S give answers back in S, and the 8-property toolkit applies to data items in S.

Proof: Zero is in S because $A\vec{0} = \vec{0}$ for any matrix A. To apply the subspace criterion, we verify that $\vec{z} = c_1\vec{x} + c_2\vec{y}$ belongs to S, provided $\vec{x}$ and $\vec{y}$ are in S. The details:

$$\begin{aligned}
A\vec{z} &= A(c_1\vec{x} + c_2\vec{y}) \\
&= A(c_1\vec{x}) + A(c_2\vec{y}) \\
&= c_1 A\vec{x} + c_2 A\vec{y} \\
&= c_1\vec{0} + c_2\vec{0} && \text{Because } A\vec{x} = A\vec{y} = \vec{0}, \text{ due to } \vec{x}, \vec{y} \text{ in } S. \\
&= \vec{0} && \text{Therefore, } A\vec{z} = \vec{0}, \text{ and } \vec{z} \text{ is in } S.
\end{aligned}$$

The proof is complete.

The Kernel Theorem in Plain Language

The Kernel Theorem says that a subspace criterion proof can be avoided by checking that data set $\boldsymbol{S}$, a subset of a vector space $\boldsymbol{R}^n$, is completely described by a system of homogeneous linear algebraic equations.

Applying the Kernel Theorem replaces a formal proof, because the conclusion is that $\boldsymbol{S}$ is a subspace of $\boldsymbol{R}^n$. Details, if any, amount to showing that the defining relations for $\boldsymbol{S}$ can be expressed as a system of homogeneous linear algebraic equations. One way to do this is to write the relations in matrix form $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$.

Example

Let $\boldsymbol{S}$ be the data set in $\boldsymbol{R}^3$ given as the intersection of the two planes $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}=\boldsymbol{0}$, $\boldsymbol{x}+\boldsymbol{2y}-\boldsymbol{z}=\boldsymbol{0}$. Then $\boldsymbol{S}$ is a subspace of $\boldsymbol{R}^3$ by the Kernel Theorem. The reason: linear homogeneous algebraic equations $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}=\boldsymbol{0}$, $\boldsymbol{x}+\boldsymbol{2y}-\boldsymbol{z}=\boldsymbol{0}$ completely describe $\boldsymbol{S}$. By the Kernel Theorem, $\boldsymbol{S}$ is a subspace of $\boldsymbol{R}^3$. Presentation details might, alternatively, define matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and verify $\boldsymbol{S}$ is the kernel of $\boldsymbol{A}$.

Not a Subspace Theorem

Theorem 2 (Testing S not a Subspace)
Let V be an abstract vector space and assume S is a subset of V. Then S is not a subspace of V provided one of the following holds.

(1) The vector $\vec{0}$ is not in S.

(2) Some $\vec{x}$ and $-\vec{x}$ are not both in S.

(3) Vector $\vec{x} + \vec{y}$ is not in S for some $\vec{x}$ and $\vec{y}$ in S.

Proof: The theorem is justified from the *Subspace Criterion*.
1. The criterion requires $\vec{0}$ is in S.
2. The criterion demands $c\vec{x}$ is in S for all scalars c and all vectors $\vec{x}$ in S.
3. According to the subspace criterion, the sum of two vectors in S must be in S.

Definition of Independence and Dependence

A list of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a vector space V are said to be **independent** provided every linear combination of these vectors is uniquely represented. **Dependent** means **not independent**.

Unique representation

An equation

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$$

implies matching coefficients: $a_1 = b_1, \ldots, a_k = b_k$.

Independence Test

Form the system in unknowns $c_1, \ldots, c_k$

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Solve for the unknowns [how to do this depends on V]. Then the vectors are independent if and only if the unique solution is $c_1 = c_2 = \cdots = c_k = 0$.

Independence test for two vectors $\vec{v}_1, \vec{v}_2$

In an abstract vector space V, form the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}.$$

Solve this equation for the two constants c_1, c_2.

Then $\vec{v}_1$, $\vec{v}_2$ are independent in V if and only if the system has unique solution $c_1 = c_2 = 0$.

There is no algorithm for how to do this, because it depends on the vector space V and sometimes on detailed information obtained from bursting the data packages $\vec{v}_1, \vec{v}_2$. If V is some R^n, then combo-swap-mult sequences apply.

Geometry and Independence

- One fixed vector is independent if and only if it is nonzero.
- Two fixed vectors are independent if and only if they form the edges of a parallelogram of positive area.
- Three fixed vectors are independent if and only if they are the edges of a parallelepiped of positive volume.

In an abstract vector space $\boldsymbol{V}$, one vector [one data package] is independent if and only if it is a nonzero vector. Two vectors [two data packages] are independent if and only if one is not a scalar multiple of the other. There is no simple test for three vectors.

Illustration

Vectors $\vec{\boldsymbol{v}}_1 = \mathbf{cos}\,\boldsymbol{x}$ and $\vec{\boldsymbol{v}}_2 = \mathbf{sin}\,\boldsymbol{x}$ are two data packages [graphs] in the vector space $\boldsymbol{V}$ of continuous functions. They are independent because one graph is not a scalar multiple of the other graph.

An Illustration of the Independence Test

Two column vectors are tested for independence by forming the system of equations $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, e.g,

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system $A\vec{c} = \vec{0}$ with

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The system $A\vec{c} = \vec{0}$ can be solved for $\vec{c}$ by combo-swap-mult methods. Because $\mathbf{rref}(A) = I$, then $c_1 = c_2 = 0$, which verifies independence.
If the system $A\vec{c} = \vec{0}$ is square, then $\det(A) \neq 0$ applies to test independence.
Determinants cannot be used directly when the system is not square, e.g., consider the homogeneous system

$$c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has vector-matrix form $A\vec{c} = \vec{0}$ with 3×2 matrix A, for which $\det(A)$ is undefined.

Rank Test

In the vector space $\boldsymbol{R}^n$, the independence test leads to a system of $\boldsymbol{n}$ linear homogeneous equations in $\boldsymbol{k}$ variables $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$. The test requires solving a matrix equation $\boldsymbol{A}\vec{\boldsymbol{c}} = \vec{\mathbf{0}}$. The signal for independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. To justify the various statements, we use the relation $\mathbf{nullity}(\boldsymbol{A}) + \mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$, where $\boldsymbol{k}$ is the column dimension of $\boldsymbol{A}$.

Theorem 3 (Rank-Nullity Test)
Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ be $\boldsymbol{k}$ column vectors in $\boldsymbol{R}^n$ and let $\boldsymbol{A}$ be the augmented matrix of these vectors. The vectors are independent if $\mathbf{rank}(\boldsymbol{A}) = \boldsymbol{k}$ and dependent if $\mathbf{rank}(\boldsymbol{A}) < \boldsymbol{k}$. The conditions are equivalent to $\mathbf{nullity}(\boldsymbol{A}) = \mathbf{0}$ and $\mathbf{nullity}(\boldsymbol{A}) > \mathbf{0}$, respectively.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $A\vec{c} = \vec{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the adjugate formula $A^{-1} = \textbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 4 (Determinant Test)
Let A be a square augmented matrix of column vectors. The column vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

Sampling Test

Let functions $f_1, \ldots, f_n$ be given and let $x_1, \ldots, x_n$ be distinct x-sample values. Define

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$. Choose $x = x_1$ and $x = x_2$ in this relation to get $A\vec{c} = \vec{0}$, where $\vec{c}$ has components c_1, c_2. If $\det(A) \neq 0$, then A^{-1} exists, and this in turn implies $\vec{c} = A^{-1}A\vec{c} = \vec{0}$. We conclude f_1, f_2 are independent.

Wronskian Test

Let functions $f_1, \ldots, f_n$ be given and let x_0 be a given point. Define

$$W = \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}.$$

Then $\det(W) \neq 0$ implies $f_1, \ldots, f_n$ are independent functions. The matrix W is called the **Wronskian matrix** of $f_1, \ldots, f_n$ and $\det(W)$ is called the **Wronskian determinant**.

Proof

We'll do the proof for $n = 2$. Details are similar for general n. Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x, $c_1 f_1(x) + c_2 f_2(x) = 0$ and $c_1 f_1'(x) + c_2 f_2'(x) = 0$. Choose $x = x_0$ in this relation to get $W\vec{c} = \vec{0}$, where $\vec{c}$ has components c_1, c_2. If $\det(W) \neq 0$, then W^{-1} exists, and this in turn implies $\vec{c} = W^{-1}W\vec{c} = \vec{0}$. We conclude f_1, f_2 are independent.

Atoms

Definition. A **base atom** is one of the functions

$$1, \quad e^{ax}, \quad \cos bx, \quad \sin bx, \quad e^{ax}\cos bx, \quad e^{ax}\sin bx$$

where $b > 0$. Define an **atom** for integers $n \geq 0$ by the formula

$$\textbf{atom} = x^n(\textbf{base atom}).$$

The powers 1, x, x^2, …are atoms (multiply base atom 1 by x^n). Multiples of these powers by $\cos bx$, $\sin bx$ are also atoms. Finally, multiplying all these atoms by e^{ax} expands and completes the list of atoms.

Alternatively, an **atom** is a function with coefficient 1 obtained as the real or imaginary part of the complex expression

$$x^n e^{ax}(\cos bx + i \sin bx).$$

Illustration

We show the powers $\mathbf{1}, \boldsymbol{x}, \boldsymbol{x}^2, \boldsymbol{x}^3$ are independent atoms by applying the Wronskian Test:

$$\boldsymbol{W} = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 0 & 0 & 2 & 6x_0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Then $\det(\boldsymbol{W}) = \mathbf{12} \neq \mathbf{0}$ implies the functions $\mathbf{1}, \boldsymbol{x}, \boldsymbol{x}^2, \boldsymbol{x}^3$ are linearly independent.

Subsets of Independent Sets are Independent

Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ make an independent set and consider the subset $\vec{v}_1, \vec{v}_2$. If

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

then also

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

where $c_3 = 0$. Independence of the larger set implies $c_1 = c_2 = c_3 = 0$, in particular, $c_1 = c_2 = 0$, and then $\vec{v}_1, \vec{v}_2$ are indpendent.

Theorem 5 (Subsets and Independence)

- A non-void subset of an independent set is also independent.
- Non-void subsets of dependent sets can be independent or dependent.

Atoms and Independence

Theorem 6 (Independence of Atoms)
Any list of distinct atoms is linearly independent.

Unique Representation

The theorem is used to extract equations from relations involving atoms. For instance:

$$(c_1 - c_2)\cos x + (c_1 + c_3)\sin x + c_1 + c_2 = 2\cos x + 5$$

implies

$$\begin{aligned} c_1 - c_2 &= 2, \\ c_1 + c_3 &= 0, \\ c_1 + c_2 &= 5. \end{aligned}$$

Atoms and Differential Equations

It is known that solutions of linear constant coefficient differential equations of order n and also systems of linear differential equations with constant coefficients have a general solution which is a linear combination of atoms.

- The harmonic oscillator $y'' + b^2 y = 0$ has general solution $y(x) = c_1 \cos bx + c_2 \sin bx$. This is a linear combination of the two atoms $\cos bx$, $\sin bx$.
- The third order equation $y''' + y' = 0$ has general solution $y(x) = c_1 \cos x + c_2 \sin x + c_3$. The solution is a linear combination of the independent atoms $\cos x$, $\sin x$, 1.
- The linear dynamical system $x'(t) = y(t)$, $y'(t) = -x(t)$ has general solution $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = -c_1 \sin t + c_2 \cos t$, each of which is a linear combination of the independent atoms $\cos t$, $\sin t$.