Coordinates and Change of Basis

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Basis

Definition. A **basis** of an abstract vector space V is a finite list of vectors $\vec{\mathrm{b}}_1, \ldots, \vec{\mathrm{b}}_p$ which is (1) *independent* and (2) *spans* V. Briefly, *independent and span*.

The keyword *span* means that $V = \mathbf{span}(\vec{\mathrm{b}}_1, \ldots, \vec{\mathrm{b}}_p)$, more precisely every $\vec{\mathrm{v}}$ in V can be expressed as $\vec{\mathrm{v}} = x_1\vec{\mathrm{b}}_1 + \cdots + x_p\vec{\mathrm{b}}_p$ for some constants $x_1, \ldots, x_p$.

Coordinate Map

Let $\vec{\mathrm{b}}_1, \ldots, \vec{\mathrm{b}}_p$ denote a **basis** of an abstract vector space V. The list of vectors is (1) *independent* and (2) *spans* V.

Example: Vectors $\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \vec{\mathrm{b}}_3$ defined by functions $y = 1, y = x, y = x^2$ on domain $(-\infty, \infty)$ generate a vector space $V = \mathrm{span}(\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \vec{\mathrm{b}}_3)$ as a subspace of the vector space W of all functions defined on domain $(-\infty, \infty)$. The functions are independent by the Wronskian Test, therefore they form a **basis** of V.

Definition. The **coordinate map** of basis $\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \vec{\mathrm{b}}_3$ is the linear map

$$T : V \to R^3 \quad \text{defined by} \quad T(x_1\vec{\mathrm{b}}_1 + x_2\vec{\mathrm{b}}_2 + x_3\vec{\mathrm{b}}_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Theorem. The coordinate map is one–to–one and onto. Briefly, the coordinate map is an **isomorphism**.

Example: Coordinate Map

Define vectors $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ by

$$\begin{array}{ll} \vec{\mathbf{b}}_1 : \ y = 1 & \text{domain } (-\infty, \infty) \\ \vec{\mathbf{b}}_2 : \ y = x & \text{domain } (-\infty, \infty) \\ \vec{\mathbf{b}}_3 : \ y = x^2 & \text{domain } (-\infty, \infty) \end{array}$$

Define vector space $V = \mathbf{span}(\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3)$ as a subspace of the vector space W of all functions defined on domain $(-\infty, \infty)$.

Independence. The Wronskian Test $\det \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{pmatrix} \neq 0$ implies the three functions are independent, therefore they form a **basis** of V.

Example: Coordinate Map, continued

Definition. The **coordinate map** of basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$ is the linear map

$$T : V \to R^3 \quad \text{defined by} \quad T(x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The coordinate map for basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$ can be written succinctly as

$$T(c_1 + c_2 x + c_3 x^2) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

EXAMPLE. Find $T\left(2 - 3x + (1+x)^2\right)$

Solution. First, $2 - 3x + (1+x)^2 = 2 - 3x + 1 + 2x + x^2 = 3 - x + x^2$, therefore

$$T\left(2 - 3x + (1+x)^2\right) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Independence-Dependence Test using a Coordinate Map

Theorem. Let $T : V_1 \to V_2$ be a linear one–to–one and onto map between vector spaces V_1 and V_2. Then T maps independent sets into independent sets and dependent sets into dependent sets.

Proof: Let $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ be an independent set in V_1 and define $\vec{\mathbf{w}}_1 = T(\vec{\mathbf{v}}_1), \ldots, \vec{\mathbf{w}}_p = T(\vec{\mathbf{v}}_p)$. We show $\vec{\mathbf{w}}_1, \ldots, \vec{\mathbf{w}}_p$ is an independent set.

Solve the equation $c_1\vec{\mathbf{w}}_1 + \cdots + c_p\vec{\mathbf{w}}_p = \vec{0}$ for constants $c_1, \ldots, c_p$ as follows.

$$\begin{array}{ll}
c_1\vec{\mathbf{w}}_1 + \cdots + c_p\vec{\mathbf{w}}_p = \vec{0} & \\
c_1 T(\vec{\mathbf{v}}_1) + \cdots + c_p T(\vec{\mathbf{v}}_p) = \vec{0} & \text{Insert definitions} \\
T(c_1\vec{\mathbf{v}}_1 + \cdots + c_p\vec{\mathbf{v}}_p) = \vec{0} & \text{linearity of } T
\end{array}$$

Because T is one–to–one, then any relation $T(\vec{\mathbf{u}}) = \vec{0}$ implies $\vec{\mathbf{u}} = \vec{0}$, giving

$$c_1\vec{\mathbf{v}}_1 + \cdots + c_p\vec{\mathbf{v}}_p = \vec{0}.$$

Because $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ is an independent set, then $c_1 = \cdots = c_p = 0$. This proves $\vec{\mathbf{w}}_1, \ldots, \vec{\mathbf{w}}_p$ is an independent set. The rest of the claims in the thoerem are proved similarly, using that fact that T has an inverse T^{-1} wich is also one–to–one and onto.

Change of Basis

Let $\vec{\mathrm{b}}_1, \ldots, \vec{\mathrm{b}}_p$ denote a **basis** of an abstract vector space V. The list of vectors is (1) *independent* and (2) *spans* V. The index p is the **dimension** of vector space V.

Let $\vec{\mathrm{c}}_1, \ldots, \vec{\mathrm{c}}_p$ denote a **second basis** of the abstract vector space V.

Definition. The coordinate maps for each basis are

$$T : V \to R^p, \quad S : V \to R^p$$

defined by the relations

$$T(x_1\vec{\mathrm{b}}_1 + \cdots + x_p\vec{\mathrm{b}}_p) = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad S(y_1\vec{\mathrm{c}}_1 + \cdots + y_p\vec{\mathrm{c}}_p) = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}.$$

The **plan** is to develop a computer method to find the weights $y_1, \ldots, y_p$ given the original weights $x_1, \ldots, x_p$. The solution is the construction of a matrix A of numbers which computes the change of basis weights by the matrix multiply equation

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}.$$

Change of Basis Matrix

Let $\vec{\mathrm{b}}_1, \ldots, \vec{\mathrm{b}}_p$ denote a **basis** of an abstract vector space V. The list of vectors is (1) *independent* and (2) *spans* V. The index p is the **dimension** of vector space V.

Let $\vec{\mathrm{c}}_1, \ldots, \vec{\mathrm{c}}_p$ denote a **second basis** of the abstract vector space V.

Definition. The **coordinate map** for the second basis is a linear map

$$S : V \to R^p$$

defined by the relation

$$S(y_1\vec{\mathrm{c}}_1 + \cdots + y_p\vec{\mathrm{c}}_p) = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}.$$

Definition. The **Change of Basis Matrix** for first basis to the the second basis is the $p \times p$ augmented matrix of column vectors

$$A = \langle S(\vec{\mathrm{b}}_1) | S(\vec{\mathrm{b}}_2) | \cdots | S(\vec{\mathrm{b}}_p) \rangle.$$

Change of Basis Matrix: Details

Let $\vec{\mathrm{v}}$ be any vector in V. Then $\vec{\mathrm{v}} = x_1\vec{\mathrm{b}}_1 + \cdots + x_p\vec{\mathrm{b}}_p$ for unique weights $x_1, \ldots, x_p$ and $T(\vec{\mathrm{v}}) = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$. Compute the matrix product as follows

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = x_1 S(\vec{\mathrm{b}}_1) + \cdots + x_p S(\vec{\mathrm{b}}_p) = S(x_1 b_1 + \cdots + x_p\vec{\mathrm{b}}_p) = S(\vec{\mathrm{v}}).$$

The computation means that $AT(\vec{\mathrm{v}}) = S(\vec{\mathrm{v}})$ or that $AT = S$ where we think of A as a linear transformation. Because $\vec{\mathrm{v}} = y_1\vec{\mathrm{c}}_1 + \cdots + y_p\vec{\mathrm{c}}_p$ for some unique weights $y_1, \ldots, y_p$, then the computation means

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}.$$

Example: Change of Basis Matrix

Define vectors $\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \vec{\mathrm{b}}_3$ by

$$\begin{array}{lll} \vec{\mathrm{b}}_1: & y = 1 & \text{domain } (-\infty, \infty) \\ \vec{\mathrm{b}}_2: & y = x & \text{domain } (-\infty, \infty) \\ \vec{\mathrm{b}}_3: & y = x^2 & \text{domain } (-\infty, \infty) \end{array}$$

Let $V = \mathrm{span}(\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \vec{\mathrm{b}}_3)$, a subspace of the vector space W of all functions on $(-\infty, \infty)$. We know that these vectors are a **basis** for V with coordinate map $T(a_1 + a_2 x + a_3 x^2) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Define a second set of vectors $\vec{\mathrm{c}}_1, \vec{\mathrm{c}}_2, \vec{\mathrm{c}}_3$ by

$$\begin{array}{lll} \vec{\mathrm{c}}_1: & y = 1 + x & \text{domain } (-\infty, \infty) \\ \vec{\mathrm{c}}_2: & y = 2 + x & \text{domain } (-\infty, \infty) \\ \vec{\mathrm{c}}_3: & y = 3 + x^2 & \text{domain } (-\infty, \infty) \end{array}$$

The isomorphism theorem applies to show that $\vec{\mathrm{c}}_1, \vec{\mathrm{c}}_2, \vec{\mathrm{c}}_3$ are independent, therefore they also span V and are a second basis for V. Details:

$$T(\vec{\mathrm{c}}_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, T(\vec{\mathrm{c}}_2) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, T(\vec{\mathrm{c}}_3) = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 0.$$

Example: Change of Basis Matrix: continued

We compute the change of basis matrix for basis $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ to basis $\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \vec{\mathbf{c}}_3$ using the coordinate map

$$\boldsymbol{S}(y_1\vec{\mathbf{c}}_1 + y_2\vec{\mathbf{c}}_2 + y_3\vec{\mathbf{c}}_3) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

as follows:

$$\boldsymbol{S}(\vec{\mathbf{b}}_1) = \boldsymbol{S}(1) = \boldsymbol{S}(\vec{\mathbf{c}}_2 - \vec{\mathbf{c}}_1) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{S}(\vec{\mathbf{b}}_2) = \boldsymbol{S}(x) = \boldsymbol{S}(2\vec{\mathbf{c}}_1 - \vec{\mathbf{c}}_2) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{S}(\vec{\mathbf{b}}_3) = \boldsymbol{S}(x^2) = \boldsymbol{S}(-3(\vec{\mathbf{c}}_2 - \vec{\mathbf{c}}_1) + \vec{\mathbf{c}}_3) = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}.$$

Then

$$\boldsymbol{A} = \text{augmented matrix of answers} = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

How to quickly recover the formula for A, the change of basis matrix

Let $p = 3$. Recovery for any p is similar. Recovery time is less than 1 minute.

Given one–to–one mappings $T : V \to R^3$ and $S : V \to R^3$, which are the coordinate maps for bases $\vec{b}_1, \vec{b}_2, \vec{b}_3$ and $\vec{c}_1, \vec{c}_2, \vec{c}_3$ respectively, then the mapping ST^{-1} is defined as a mapping from R^3 to R^3. This mapping is linear, one–to–one and onto. The theory of linear transformations from R^n into R^m provides for some 3×3 matrix A a matrix multiply identity

$$ST^{-1}(\vec{x}) = A\vec{x}$$

Let $\vec{x} = T(\vec{v})$, possible because T is onto. The identity becomes

$$S(\vec{v}) = AT(\vec{v}).$$

To find the columns of the matrix A, first replace $T(\vec{v})$ by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which simply means $\vec{v} = \vec{b}_1$. Then column 1 of A is $S(\vec{b}_1)$. Repeat by replacing $T(\vec{v})$ successively by the remaining columns of the identity matrix to determine the remaining columns of A as $S(\vec{b}_2)$, $S(\vec{b}_3)$. Then

$$A = \langle S(\vec{b}_1) | S(\vec{b}_2) | S(\vec{b}_3) \rangle.$$