Determinant Theory

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Unique Solution of a $\mathbf{2 \times 2}$ System

The $\mathbf{2 \times 2}$ system

$$\begin{array}{rcrcl} \boldsymbol{ax} & + & \boldsymbol{by} & = & \boldsymbol{e}, \\ \boldsymbol{cx} & + & \boldsymbol{dy} & = & \boldsymbol{f}, \end{array} \tag{1}$$

has a unique solution provided $\boldsymbol{\Delta = ad - bc}$ is nonzero, in which case the solution is given by

$$\boldsymbol{x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}.} \tag{2}$$

This result is called **Cramer's Rule** for $\mathbf{2 \times 2}$ systems, learned in college algebra.

Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad |\boldsymbol{A}| = \begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix}.$$

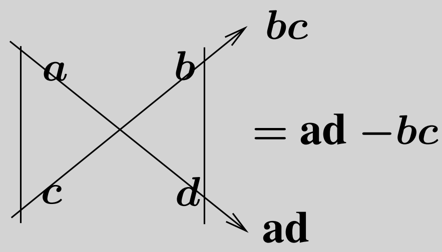

Figure 1. Sarrus' $\mathbf{2} \times \mathbf{2}$ rule. A diagram for $|\boldsymbol{A}| = \mathbf{ad} - \mathrm{bc}$.

The boldface product $\mathbf{ad}$ is the product of the main diagonal entries and the other product $\boldsymbol{bc}$ is from the anti-diagonal. Memorize as **down arrows minus up arrows**.
Cramer's $\mathbf{2} \times \mathbf{2}$ rule in determinant notation is

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{3}$$

Relation to Inverse Matrices

System

$$\begin{array}{rcrcl} ax & + & by & = & e, \\ cx & + & dy & = & f, \end{array} \tag{4}$$

can be expressed as the vector-matrix system $A\vec{u} = \vec{b}$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

Inverse matrix theory implies

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \vec{u} = A^{-1}\vec{b} = \frac{1}{ad - bc}\begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.$$

Cramer's Rule is a compact summary of the unique solution of system (4).

Unique Solution of an $n \times n$ System

System

(5)

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

can be written as an $n \times n$ vector-matrix equation $A\vec{\mathrm{x}} = \vec{\mathrm{b}}$, where $\vec{\mathrm{x}} = (x_1, \ldots, x_n)$ and $\vec{\mathrm{b}} = (b_1, \ldots, b_n)$. The system has a unique solution provided the **determinant of coefficients** $\Delta = |A|$ is nonzero, and then **Cramer's Rule** for $n \times n$ systems gives

(6)

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta}.$$

Symbol $\Delta_j = |B|$, where matrix B has the same columns as matrix A, except $\mathrm{col}(B, j) = \vec{\mathrm{b}}$.

Determinants of Order n

Determinants for $n \times n$ matrices will be defined shortly; intuition from the 2×2 case and Sarrus' rule should suffice for the moment.

Determinant Notation for Cramer's Rule

The **determinant of coefficients** for system $A\vec{x} = \vec{b}$ is denoted by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{7}$$

The other n determinants in Cramer's rule (6) are given by

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}. \tag{8}$$

Important: Use vertical bars for determinants. Use parentheses or brackets for matrices.

College Algebra Definition of Determinant

Given an $\boldsymbol{n} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$, define

$$|\boldsymbol{A}| = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)}\, a_{1\sigma_1} \cdots a_{n\sigma_n}. \tag{9}$$

Formula explained:

- Symbol a_{ij} denotes the element in row i and column j of the matrix A.
- The symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ stands for a rearrangement of the subscripts 1, 2, ..., n.
- Symbol S_n is the set of all possible rearrangements σ.
- Nonnegative integer parity(σ) is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers σ_1, ..., σ_n into natural order 1, ..., n.

College Algebra Definition and Sarrus' 3×3 Rule

For a $\mathbf{3 \times 3}$ matrix, the College Algebra formula reduces to **Sarrus' 3×3 Rule**

$$
\begin{aligned}
|A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\
&\quad - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}.
\end{aligned} \tag{10}
$$

Diagram for Sarrus' 3×3 Rule

The number $|A|$ in the 3×3 case **only** can be computed by the algorithm in Figure 2, which parallels the one for 2×2 matrices. The 5×3 array is made by copying the first two rows of A into rows 4 and 5.

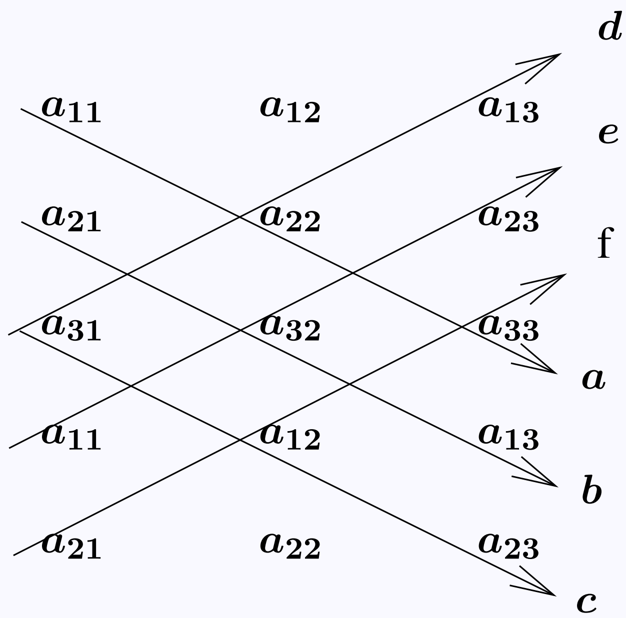

Figure 2. Sarrus' rule diagram for 3×3 matrices, which gives $|A| = (a + b + c) - (d + e + f)$. Memorize as **down minus up**

Warning: *There is no Sarrus' rule diagram for* 4×4 *or larger matrices*!

Transpose Rule

A consequence of the college algebra definition of determinant is the relation

$$|\boldsymbol{A}| = |\boldsymbol{A}^T|$$

where $\boldsymbol{A}^T$ means the transpose of $\boldsymbol{A}$, obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.

How to Compute the Value of any Determinant

- **Four Rules**. These are the *Triangular Rule*, *Combination Rule*, *Multiply Rule* and the *Swap Rule*.
- **Special Rules**. These apply to evaluate a determinant as zero.
- **Cofactor Expansion**. This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.
- **Hybrid Method**. The four rules and the cofactor expansion are combined.

Four Rules

Triangular The value of $|A|$ for either an upper triangular or a lower triangular matrix A is the product of the diagonal elements:

$$|A| = a_{11}a_{22}\cdots a_{nn}.$$

This is a one-arrow Sarrus' rule.

Swap If B results from A by swapping two rows, then

$$|A| = (-1)|B|.$$

Combination The value of $|A|$ is unchanged by adding a multiple of a row to a different row.

Multiply If one row of A is multiplied by constant c to create matrix B, then

$$|B| = c|A|.$$

1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

$$\begin{vmatrix} \mathbf{12} & \mathbf{6} & \mathbf{0} \\ \mathbf{11} & \mathbf{5} & \mathbf{1} \\ \mathbf{10} & \mathbf{2} & \mathbf{2} \end{vmatrix} = \mathbf{24.}$$

Solution: Let D denote the value of the determinant. Then

$$D = \begin{vmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{vmatrix}$$ Given.

$$= \begin{vmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{vmatrix}$$ `combo(1,2,-1), combo(1,3,-1)`. Combination leaves the determinant unchanged.

$$= 6\begin{vmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{vmatrix}$$ Multiply rule $m = 1/6$ on row 1 factors out a 6.

$$= 6\begin{vmatrix} 0 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{vmatrix}$$ `combo(1,3,1), combo(2,1,2)`.

$$= -6\begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{vmatrix}$$ `swap(1,2)`. Swap changes the sign of the determinant.

$$= 6\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{vmatrix}$$ Multiply rule $m = -1$ on row 1.

$$= 6\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{vmatrix}$$ `combo(2,3,-3)`.

$= 6(1)(-1)(-4) = 24$ Triangular rule. Formula verified.

Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

Triangular The value of $|A|$ for either an upper triangular or a lower triangular matrix A is the product of the diagonal elements: $|A| = a_{11}a_{22}\cdots a_{nn}$. This is a one-arrow Sarrus' rule valid for dimension n.

Swap If E is an elementary matrix for a swap rule, then $|EA| = (-1)|A|$.

Combination If E is an elementary matrix for a combination rule, then $|EA| = |A|$.

Multiply If E is an elementary matrix for a multiply rule with multiplier $m \neq 0$, then $|EA| = m|A|$.

Because $|E| = 1$ for a combination rule, $|E| = -1$ for a swap rule and $|E| = c$ for a multiply rule with multiplier $c \neq 0$, it follows that for any elementary matrix E there is the **determinant multiplication rule**

$$|EA| = |E||A|.$$

Special Determinant Rules

The results are stated for rows but also hold for columns, because $|\boldsymbol{A}| = |\boldsymbol{A}^T|$.

Zero row	If one row of $\boldsymbol{A}$ is zero, then $\lvert\boldsymbol{A}\rvert = \boldsymbol{0}$.
Duplicate rows	If two rows of $\boldsymbol{A}$ are identical, then $\lvert\boldsymbol{A}\rvert = \boldsymbol{0}$.
RREF $\neq \boldsymbol{I}$	If $\mathbf{rref}(\boldsymbol{A}) \neq \boldsymbol{I}$, then $\lvert\boldsymbol{A}\rvert = \boldsymbol{0}$.
Common factor	The relation $\lvert\boldsymbol{A}\rvert = \boldsymbol{c}\lvert\boldsymbol{B}\rvert$ holds, provided $\boldsymbol{A}$ and $\boldsymbol{B}$ differ only in one row, say row $\boldsymbol{j}$, for which $\mathbf{row}(\boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{c}\,\mathbf{row}(\boldsymbol{B}, \boldsymbol{j})$.
Row linearity	The relation $\lvert\boldsymbol{A}\rvert = \lvert\boldsymbol{B}\rvert + \lvert\boldsymbol{C}\rvert$ holds, provided $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ differ only in one row, say row $\boldsymbol{j}$, for which $\mathbf{row}(\boldsymbol{A}, \boldsymbol{j}) = \mathbf{row}(\boldsymbol{B}, \boldsymbol{j}) + \mathbf{row}(\boldsymbol{C}, \boldsymbol{j})$.

Cofactor Expansion for 3×3 Matrices

This is a review the college algebra topic, where the dimension of $\boldsymbol{A}$ is $\mathbf{3}$.

Cofactor row expansion means the following formulas are valid:

$$\begin{aligned}
|\boldsymbol{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{21}(-1)\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1)\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1)\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{31}(+1)\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1)\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1)\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\end{aligned}$$

The formulas expand a $\mathbf{3} \times \mathbf{3}$ determinant in terms of $\mathbf{2} \times \mathbf{2}$ determinants, along a row of $\boldsymbol{A}$. The attached signs ± 1 are called the **checkerboard signs**, to be defined shortly. The $\mathbf{2} \times \mathbf{2}$ determinants are called **minors** of the $\mathbf{3} \times \mathbf{3}$ determinant $|\boldsymbol{A}|$. The checkerboard sign together with a minor is called a **cofactor**.

Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the 2×2 determinants in the expansion. To illustrate, row 1 cofactor expansion gives

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 7 \\ 5 & 4 & 8 \end{vmatrix} &= 3(+1)\begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 3(+1)(8-28)+0+0 \\ &= -60. \end{aligned}$$

What has been said for rows also applies to columns, due to the transpose formula

$$|A| = \left|A^T\right|.$$

Minor

The $(n-1)\times(n-1)$ determinant obtained from $|A|$ by striking out row i and column j is called the (i,j)–minor of A and denoted $\textsf{minor}(A,i,j)$. Literature might use M_{ij} for a minor.

Cofactor

The (i,j)–cofactor of A is $\textsf{cof}(A,i,j) = (-1)^{i+j}\,\textsf{minor}(A,i,j)$.
Multiplicative factor $(-1)^{i+j}$ is called the **checkerboard sign**, because its value can be determined by counting *plus*, *minus*, *plus*, etc., from location $(1,1)$ to location (i,j) in any checkerboard fashion.

Expansion of Determinants by Cofactors

$$|A| = \sum_{j=1}^{n} a_{kj}\,\textsf{cof}(A,k,j), \quad |A| = \sum_{i=1}^{n} a_{i\ell}\,\textsf{cof}(A,i,\ell), \tag{11}$$

In (11), $1 \le k \le n$, $1 \le \ell \le n$. The first expansion is called a **cofactor row expansion** and the second is called a **cofactor column expansion**. The value $\textsf{cof}(A,i,j)$ is the cofactor of element a_{ij} in $|A|$, that is, the checkerboard sign times the minor of a_{ij}.

2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity

$$\begin{vmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{vmatrix} = 5(6a - b).$$

Solution: Let D denote the value of the determinant. Then

$$D = \begin{vmatrix} 10 & 5 & 0 \\ 11 & 5 & a \\ 10 & 2 & b \end{vmatrix}$$ Given.

$$= \begin{vmatrix} 10 & 5 & 0 \\ 1 & 0 & a \\ 0 & -3 & b \end{vmatrix}$$ Combination leaves the determinant unchanged: `combo(1,2,-1), combo(1,3,-1)`.

$$= \begin{vmatrix} 0 & 5 & -10a \\ 1 & 0 & a \\ 0 & -3 & b \end{vmatrix}$$ `combo(2,1,-10)`.

$$= (1)(-1)\begin{vmatrix} 5 & -10a \\ -3 & b \end{vmatrix}$$ Cofactor expansion on column 1.

$$= (1)(-1)(5b - 30a)$$ Sarrus' rule for $n = 2$.

$$= 5(6a - b).$$ Formula verified.

3 Example (Cramer's Rule) Solve by Cramer's rule the system of equations

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & - & x_4 & = & 1, \\ x_1 & + & x_2 & - & & & x_4 & = & -1, \\ & & 3x_2 & + & x_3 & + & x_4 & = & 3, \\ x_1 & + & & & x_3 & - & x_4 & = & 0, \end{array}$$

verifying $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

Solution: Form the four determinants $\Delta_1, \ldots, \Delta_4$ from the base determinant Δ as follows:

$$\Delta = \begin{vmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{vmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers $\Delta = -2$, $\Delta_1 = -2$, $\Delta_2 = 0$, $\Delta_3 = -2$, $\Delta_4 = -4$, therefore Cramer's rule implies the solution

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.$$

Then $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

Maple Code for Cramer's Rule

The details of the computation above can be checked in computer algebra system `maple` as follows.

```
with(LinearAlgebra):
A:=Matrix([
[2, 3, 1, -1], [1, 1, 0, -1],
[0, 3, 1, 1], [1, 0, 1, -1]]);
Delta:= Determinant(A);
b:=Vector([1,-1,3,0]):
B1:=A: Column(B1,1):=b:
Delta1:=Determinant(B1);
x[1]:=Delta1/Delta;
LinearSolve(A,b);
```

The Adjugate Matrix

The **adjugate** $\mathbf{adj}(A)$ of an $n \times n$ matrix A is the transpose of the matrix of cofactors,

$$\mathbf{adj}(A) = \begin{pmatrix} \mathbf{cof}(A,1,1) & \mathbf{cof}(A,1,2) & \cdots & \mathbf{cof}(A,1,n) \\ \mathbf{cof}(A,2,1) & \mathbf{cof}(A,2,2) & \cdots & \mathbf{cof}(A,2,n) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{cof}(A,n,1) & \mathbf{cof}(A,n,2) & \cdots & \mathbf{cof}(A,n,n) \end{pmatrix}^T .$$

A cofactor $\mathbf{cof}(A,i,j)$ is the checkerboard sign $(-1)^{i+j}$ times the corresponding minor determinant $\mathbf{minor}(A,i,j)$.

Adjugate of a 2×2

$$\mathbf{adj}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In words: *swap the diagonal elements and change the sign of the off–diagonal elements.*

Adjugate Formula for the Inverse

For any $n \times n$ matrix

$$A \cdot \textbf{adj}(A) = \textbf{adj}(A) \cdot A = |A|\, I.$$

The equation is valid even if A is not invertible. The relation suggests several ways to find $|A|$ from A and $\textbf{adj}(A)$ with one dot product.

For an invertible matrix A, the relation implies $A^{-1} = \textbf{adj}(A)/|A|$:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} \textbf{cof}(A,1,1) & \textbf{cof}(A,1,2) & \cdots & \textbf{cof}(A,1,n) \\ \textbf{cof}(A,2,1) & \textbf{cof}(A,2,2) & \cdots & \textbf{cof}(A,2,n) \\ \vdots & \vdots & \cdots & \vdots \\ \textbf{cof}(A,n,1) & \textbf{cof}(A,n,2) & \cdots & \textbf{cof}(A,n,n) \end{pmatrix}^T$$

Application: Adjugate Shortcut

Given $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, then we can compute $\textsf{adj}(\boldsymbol{A}) = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix}$.

Suppose that we mark some unknown entries in $\textsf{adj}(\boldsymbol{A})$ by ? and write $|\boldsymbol{A}|$ for the determinant of $\boldsymbol{A}$. Then the formula $\boldsymbol{A}\,\textsf{adj}(\boldsymbol{A}) = \textsf{adj}(\boldsymbol{A})\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{I}$ becomes

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} = \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} |\boldsymbol{A}| & 0 & 0 \\ 0 & |\boldsymbol{A}| & 0 \\ 0 & 0 & |\boldsymbol{A}| \end{pmatrix}.$$

While the second product $\textsf{adj}(\boldsymbol{A})\boldsymbol{A}$ contains useless information, the first product gives $\mathbf{row}(\boldsymbol{A}, 2)\,\mathbf{col}(\textsf{adj}(\boldsymbol{A}), 2) = |\boldsymbol{A}|$. Because the values are known, then $|\boldsymbol{A}| = 6 + 1 + 0 = 7$.

Knowing A and $\textsf{adj}(A)$ gives the value of $|A|$ in one dot product.

Elementary Matrices

Theorem 1 (Determinants and Elementary Matrices)
Let $\boldsymbol{E}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ elementary matrix. Then

Combination	$\lvert\boldsymbol{E}\rvert = \mathbf{1}$
Multiply	$\lvert\boldsymbol{E}\rvert = \boldsymbol{m}$ for multiplier $\boldsymbol{m}$.
Swap	$\lvert\boldsymbol{E}\rvert = -\mathbf{1}$
Product	$\lvert\boldsymbol{EX}\rvert = \lvert\boldsymbol{E}\rvert\lvert\boldsymbol{X}\rvert$ for all $\boldsymbol{n} \times \boldsymbol{n}$ matrices $\boldsymbol{X}$.

Theorem 2 (Determinants and Invertible Matrices)
Let $\boldsymbol{A}$ be a given invertible matrix. Then

$$\lvert\boldsymbol{A}\rvert = \frac{(-1)^{s}}{m_1 m_2 \cdots m_r}$$

where s is the number of swap rules applied and $m_1, m_2, \ldots, m_r$ are the nonzero multipliers used in multiply rules when $\boldsymbol{A}$ is reduced to $\mathbf{rref}(\boldsymbol{A})$.

Determinant Products

Theorem 3 (Determinant Product Rule)

Let A and B be given $n \times n$ matrices. Then

$$|AB| = |A||B|.$$

Proof

Assume A^{-1} does not exist. Then A has zero determinant, which implies $|AB| = 0$. If $|B| = 0$, then $B\vec{x} = \vec{0}$ has infinitely many solutions, in particular a nonzero solution $\vec{x}$. Multiply $B\vec{x} = \vec{0}$ by A, then $AB\vec{x} = \vec{0}$ which implies AB is not invertible. Then the identity $|AB| = |A||B|$ holds, because both sides are zero. If $|B| \neq 0$ but $|A| = 0$, then there is a nonzero $\vec{y}$ with $A\vec{y} = \vec{0}$. Define $\vec{x} = B^{-1}\vec{y}$. Then $AB\vec{x} = A\vec{y} = \vec{0}$, with $\vec{x} \neq \vec{0}$, which implies AB is not invertible, and as earlier in this paragraph, the identity holds. This completes the proof when A is not invertible.

Assume A is invertible. In particular, $\text{rref}(A^{-1}) = I$. Write $I = \text{rref}(A^{-1}) = E_1E_2 \cdots E_kA^{-1}$ for elementary matrices $E_1, \ldots, E_k$. Then $A = E_1E_2 \cdots E_k$ and

$$AB = E_1E_2 \cdots E_kB. \tag{12}$$

The theorem follows from repeated application of the basic identity $|EX| = |E||X|$ to relation (12), because

$$|AB| = |E_1| \cdots |E_k||B| = |A||B|.$$