Basis, Dimension, Kernel, Image

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- Main Results: Rank-Nullity, Row Rank, Pivot Method
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Definitions: Pivot and Basis

Pivot of A	A column in matrix A is called a **pivot column of A** provided the corresponding column in $\mathrm{rref}(A)$ contains a leading one.
Basis of V	It is an independent set $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ from data set V whose linear combinations generate all data items in V. Briefly: the vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ are independent and span V.

Definitions: Rank and Nullity

$\mathbf{rank}(A)$	The number of leading ones in $\mathrm{rref}(A)$
$\mathbf{nullity}(A)$	The number of columns of A minus $\mathrm{rank}(A)$

Main Results: Dimension, Pivot Theorem

Theorem 1 (Dimension)
If a vector space V has an independent spanning set $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ and another independent spanning set $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_q$, then $p = q$. The **dimension** of V is this unique number p. We write $p = \dim(V)$.

Theorem 2 (The Pivot Theorem)

- The pivot columns of a matrix A are linearly independent.
- A non-pivot column of A is a linear combination of the pivot columns of A.

The proofs can be found in web documents and also in the textbook by Edwards and Penny or in David Lay's textbook. Self-contained proofs of the statements of the pivot theorem appear later in these slides.

Lemma 1 Let B be invertible and $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ independent. Then $B\vec{\mathbf{v}}_1, \ldots, B\vec{\mathbf{v}}_p$ are independent.

Proof of Independence of the Pivot Columns

Consider the fundamental toolkit sequence identity $\mathbf{rref}(A) = EA$ where $E = E_k \cdots E_2 E_1$ is a product of elementary matrices. Let $B = E^{-1}$. Then

$$\mathbf{col}(\mathbf{rref}(A), j) = E\,\mathbf{col}(A, j)$$

implies that a pivot column j of A satisfies

$$\mathbf{col}(A, j) = B\,\mathbf{col}(I, j).$$

Because the columns of I are independent, then also the pivot columns of A are independent, by the Lemma.

Proof of Non-Pivot Column Dependence

Using matrix B from the previous proof, $\vec{u} = B\vec{v}$ holds for a non-pivot column $\vec{u}$ of A and its corresponding non-pivot column $\vec{v}$ in $C = \mathbf{rref}(A)$. Because each nonzero row of C has a leading one, if a component $v_i \neq 0$, then row i of C has a leading one in column $j_i < i$. Then $\mathbf{col}(C, j_i)$ is a column of the identity I and

$$\vec{v} = \sum_{v_i \neq 0} v_i \, \mathbf{col}(C, j_i).$$

Multiply the preceding display by B to give

$$\begin{aligned} \vec{u} &= B\vec{v} \\ &= \sum_{v_i \neq 0} v_i B \, \mathbf{col}(C, j_i) \\ &= \sum_{v_i \neq 0} v_i \, \mathbf{col}(A, j_i). \end{aligned}$$

Then $\vec{u}$ is a linear combination of pivot columns of A.

Main Results: Rank-Nullity, Row Rank, Pivot Method

Theorem 3 (Rank-Nullity Equation)
$\mathbf{rank}(\boldsymbol{A}) + \mathbf{nullity}(\boldsymbol{A}) =$ column dimension of $\boldsymbol{A}$

Theorem 4 (Row Rank Equals Column Rank)
The number of independent rows of a matrix $\boldsymbol{A}$ equals the number of independent columns of $\boldsymbol{A}$. Equivalently, $\mathbf{rank}(\boldsymbol{A}) = \mathbf{rank}(\boldsymbol{A}^{\boldsymbol{T}})$.

Theorem 5 (Pivot Method)
Let $\boldsymbol{A}$ be the augmented matrix of $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$. Let the leading ones in $\mathbf{rref}(\boldsymbol{A})$ occur in columns $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_p$. Then a largest independent subset of the $\boldsymbol{k}$ vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ is the set

$$\vec{\mathbf{v}}_{i_1}, \vec{\mathbf{v}}_{i_2}, \ldots, \vec{\mathbf{v}}_{i_p}.$$

Proof that $\mathbf{rank}(A) = \mathbf{rank}(A^T)$

Let S denote the set of all linear combinations of the rows of A. Then S is a subspace, known as the row space of A. A toolkit sequence from A to $\mathbf{rref}(A)$ consists of combination, swap and multiply operations on the rows of A (replace, swap and scale in David Lay's textbook). Therefore, each nonzero row of $\mathbf{rref}(A)$ is a linear combination of the rows of A. Because these rows are independent and span S, then they are a basis for S. The size of the basis is $\mathbf{rank}(A)$.

The pivot theorem applied to A^T implies that each vector in S is a linear combination of the pivot columns of A^T. Because the pivot columns of A^T are independent and span S, then they are a basis for S. The size of the basis is $\mathbf{rank}(A^T)$.

The two competing bases for S have sizes $\mathbf{rank}(A)$ and $\mathbf{rank}(A^T)$, respectively. But the size of a basis is unique, called the dimension of the subspace S, hence the equality

$$\mathbf{rank}(A) = \mathbf{rank}(A^T).$$

Definitions: Kernel, Image, rowspace, colspace

$\text{kernel}(A) = \text{nullspace}(A) = \{\vec{x} \,:\, A\vec{x} = \vec{0}\}$.
$\text{Image}(A) = \text{colspace}(A) = \{\vec{y} \,:\, \vec{y} = A\vec{x} \text{ for some } \vec{x}\}$.
$\text{rowspace}(A) = \text{colspace}(A^T) = \{\vec{w} \,:\, \vec{w} = A^T\vec{y} \text{ for some } \vec{y}\}$.

How to Compute Nullspace, Rowspace and Colspace

Null Space. Compute $\text{rref}(A)$. Write out the general solution $\vec{x}$ to $A\vec{x} = \vec{0}$, where the free variables are assigned parameter names $t_1, \ldots, t_k$. Report the basis for $\text{nullspace}(A)$ as the list $\partial_{t_1}\vec{x}, \ldots, \partial_{t_k}\vec{x}$.

Column Space. Compute $\text{rref}(A)$. Identify the pivot columns $i_1, \ldots, i_k$. Report the basis for $\text{colspace}(A)$ as the list of columns $i_1, \ldots, i_k$ of A.

Row Space. Compute $\text{rref}(A^T)$. Identify the pivot columns $j_1, \ldots, j_\ell$ of A^T. Report the basis for $\text{rowspace}(A)$ as the list of rows $j_1, \ldots, j_\ell$ of A.

Alternatively, compute $\text{rref}(A)$, then $\text{rowspace}(A)$ has a *different* basis consisting of the list of nonzero rows of $\text{rref}(A)$.

Dimension, Kernel and Image

Symbol $\dim(V)$ equals the number of elements in a basis for V.

Theorem 6 (Dimension Identities)

(a) $\dim(\text{nullspace}(A)) = \dim(\text{kernel}(A)) = \text{nullity}(A)$

(b) $\dim(\text{colspace}(A)) = \dim(\text{Image}(A)) = \text{rank}(A)$

(c) $\dim(\text{rowspace}(A)) = \text{rank}(A)$

(d) $\dim(\text{kernel}(A)) + \dim(\text{Image}(A)) =$ column dimension of A

(e) $\dim(\text{kernel}(A)) + \dim(\text{kernel}(A^T)) =$ column dimension of A

Testing Bases for Equivalence

Theorem 7 (Equivalence Test for Bases)
Define augmented matrices

$$\boldsymbol{B} = \mathbf{aug}(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k), \quad \boldsymbol{C} = \mathbf{aug}(\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_\ell), \quad \boldsymbol{W} = \mathbf{aug}(\boldsymbol{B}, \boldsymbol{C}).$$

Then relation $\boldsymbol{k} = \boldsymbol{\ell} = \mathbf{rank}(\boldsymbol{B}) = \mathbf{rank}(\boldsymbol{C}) = \mathbf{rank}(\boldsymbol{W})$ implies

1. $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ is an independent set.
2. $\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_\ell$ is an independent set.
3. $\mathbf{span}\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\} = \mathbf{span}\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_\ell\}$

In particular, $\mathbf{colspace}(\boldsymbol{B}) = \mathbf{colspace}(\boldsymbol{C})$ and each set of vectors is an equivalent basis for this vector space.

Proof: Because $\mathbf{rank}(\boldsymbol{B}) = \boldsymbol{k}$, then the first $\boldsymbol{k}$ columns of $\boldsymbol{W}$ are independent. If some column of $\boldsymbol{C}$ is independent of the columns of $\boldsymbol{B}$, then $\boldsymbol{W}$ would have $\boldsymbol{k} + \mathbf{1}$ independent columns, which violates $\boldsymbol{k} = \mathbf{rank}(\boldsymbol{W})$. Therefore, the columns of $\boldsymbol{C}$ are linear combinations of the columns of $\boldsymbol{B}$. Then vector space $\mathbf{colspace}(\boldsymbol{C})$ is a subspace of vector space $\mathbf{colspace}(\boldsymbol{B})$. Because both vector spaces have dimension $\boldsymbol{k}$, then $\mathbf{colspace}(\boldsymbol{B}) = \mathbf{colspace}(\boldsymbol{C})$. The proof is complete.

Equivalent Bases: Computer Illustration

The following `maple` code applies the theorem to verify that two bases are equivalent:

1. The basis is determined from the `ColumnSpace` command in `maple`.

2. The basis is determined from the pivot columns of $\boldsymbol{A}$.

In `maple`, the report of the column space basis is identical to the nonzero rows of $\mathbf{rref}(\boldsymbol{A}^T)$.

```
with(LinearAlgebra):
A:=Matrix([[1,0,3],[3,0,1],[4,0,0]]);
ColumnSpace(A);          # Solve Ax=0, basis v1,v2 below
v1:=<2,0,-1>;v2:=<0,2,3>; # fractions removed
ReducedRowEchelonForm(A); # Determine pivot cols=1,3
u1:=Column(A,1); u2:=Column(A,3); # pivot col basis
B:=<v1|v2>; C:=<u1|u2>;
W:=<B|C>;
Rank(B),Rank(C),Rank(W); # Test requires all equal to 2
```

A False Test for Equivalent Bases

The relation

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C})$$

holds for a substantial number of matrices $\boldsymbol{B}$ and $\boldsymbol{C}$. However, it does not imply that each column of $\boldsymbol{C}$ is a linear combination of the columns of $\boldsymbol{B}$. In particular, it is possible that $\mathbf{colspace}(\boldsymbol{B}) \neq \mathbf{colspace}(\boldsymbol{C})$.

For example, define

$$\boldsymbol{B} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Then

$$\mathbf{rref}(\boldsymbol{B}) = \mathbf{rref}(\boldsymbol{C}) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

but $\mathbf{col}(\boldsymbol{C}, \mathbf{2})$ is not a linear combination of the columns of $\boldsymbol{B}$. This means $\mathbf{colspace}(\boldsymbol{B}) \neq \mathbf{colspace}(\boldsymbol{C})$.

Geometrically, the column space of $\boldsymbol{B}$ is the span of two independent vectors, which is a plane in $\boldsymbol{R}^{\mathbf{3}}$. The column space of $\boldsymbol{C}$ is also a plane, but a different one which intersects the plane for $\boldsymbol{B}$ only along the line $\boldsymbol{L}$ determined by the two points $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{0}, \mathbf{1})$.