5.4 Independence, Span and Basis

The technical topics of independence, dependence and span apply to the study of Euclidean spaces $\mathcal{R}^2$, $\mathcal{R}^3$, ..., $\mathcal{R}^n$ and also to the continuous function space $C(E)$, the space of differentiable functions $C^1(E)$ and its generalization $C^n(E)$, and to general abstract vector spaces.

Basis and General Solution

The term **basis** has been introduced earlier for systems of linear algebraic equations. To review, a basis is obtained from the vector general solution $\vec{x}$ of matrix equation $A\vec{x} = \vec{0}$ by computing the partial derivatives ∂_{t_1}, ∂_{t_2}, ... of $\vec{x}$, where t_1, t_2, ... is the list of invented symbols assigned to the free variables identified in $\mathbf{rref}(A)$.

The partial derivatives are **special solutions** to the homogeneous equation $A\vec{x} = \vec{0}$. Knowing the special solutions is sufficient for writing out the general solution. In summary:

> A **basis** is an abbreviation or shortcut notation for the general solution.

Deeper properties have been isolated for the list of special solutions obtained from the partial derivatives $\partial_{t_1}\vec{x}$, $\partial_{t_2}\vec{x}$, ... The most important properties are **span** and **independence**.

Independence, Span and Basis

A list of vectors $\vec{v}_1$, ..., $\vec{v}_k$ is said to **span** a vector space V (definition on page 297), written

$$V = \mathbf{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k),$$

provided V consists of exactly the set of all linear combinations

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

The notion originates with the general solution $\vec{v}$ of a homogeneous matrix system $A\vec{v} = \vec{0}$, where the invented symbols t_1, ..., t_k are the constants c_1, ..., c_k and the vector partial derivative list $\partial_{t_1}\vec{v}$, ..., $\partial_{t_k}\vec{v}$ is the list of special solution vectors $\vec{v}_1$, ..., $\vec{v}_k$.

Vectors $\vec{v}_1$, ..., $\vec{v}_k$ are said to be **independent** provided each linear combination $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is represented by a unique set of constants c_1, ..., c_k. See pages 369 and 375 for independence tests.

Definition 6 (Basis)
A **basis** of a vector space V is defined to be a list of independent vectors $\vec{v}_1, \ldots, \vec{v}_k$ which spans V. A basis is tested by two checkpoints:

1. The list of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is independent.
2. The vectors span V, written $V = \mathbf{span}(\vec{v}_1, \ldots, \vec{v}_k)$.

Bases are used to express the **general solution** of a linear problem. The **spanning condition** means that every possible vector in V is a linear combination of basis elements. The **independence condition** means that linear combinations are uniquely represented, which, in practical terms, means that the general solution expression has *the fewest possible terms.*

The Vector Spaces $\mathcal{R}^n$

The vector space $\mathcal{R}^n$ of n-element fixed column vectors (or row vectors) is from the view of applications a *storage system for organization of numerical data sets* that is equipped with an algebraic toolkit. The organizational scheme induces a *data structure* onto the numerical data set. In particular, whether needed or not, there are pre-defined operations of addition (+) and scalar multiplication ($\cdot$) which apply to fixed vectors. The two operations on fixed vectors satisfy the *closure law* and in addition obey the *eight algebraic vector space properties.* We view the vector space $V = \mathcal{R}^n$ as the **data set** consisting of data item packages. The **toolkit** is the following set of eight algebraic properties.

Closure	The operations $\vec{X}+\vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V.	
Addition	$\vec{X}+\vec{Y}=\vec{Y}+\vec{X}$	commutative
	$\vec{X}+(\vec{Y}+\vec{Z})=(\vec{Y}+\vec{X})+\vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0}+\vec{X}=\vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X}+(-\vec{X})=\vec{0}$	negative
Scalar multiply	$k(\vec{X}+\vec{Y})=k\vec{X}+k\vec{Y}$	distributive I
	$(k_1+k_2)\vec{X}=k_1\vec{X}+k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X})=(k_1k_2)\vec{X}$	distributive III
	$1\vec{X}=\vec{X}$	identity

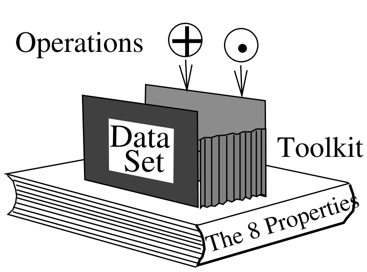

Figure 12. A Data Storage System.
A vector space is a data set of data item packages plus a storage system which organizes the data. A toolkit is provided consisting of operations + and $\cdot$ plus 8 algebraic vector space properties.

Fixed Vectors and the Toolkit

Scalar multiplication of fixed vectors is commonly used for re-scaling, especially to unit systems fps, cgs and mks. For instance, a numerical data set of lengths recorded in meters (mks) is re-scaled to centimeters (cgs) using scale factor $k = 100$.

Addition and subtraction of fixed vectors is used in a variety of calculations, which includes averages, difference quotients and calculus operations like integration.

Planar Plots and the Toolkit

The data set for a plot problem consists of the plot points in $\mathcal{R}^2$, which are the **dots** for the connect-the-dots graphic. Assume the function $y(x)$ to be plotted comes from a differential equation like $y' = f(x, y)$, then Euler's numerical method could be used for the sequence of dots in the graphic. In this case, the next dot is represented as $\vec{v}_2 = \vec{v}_1 + \vec{E}(\vec{v}_1)$. Symbol $\vec{v}_1$ is the previous dot and symbol $\vec{E}(\vec{v}_1)$ is the Euler increment. We define

$$\vec{v}_1 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \vec{E}(\vec{v}_1) = h \begin{pmatrix} 1 \\ f(x_0, y_0) \end{pmatrix},$$
$$\vec{v}_2 = \vec{v}_1 + \vec{E}(\vec{v}_1) = \begin{pmatrix} x_0 + h \\ y_0 + hf(x_0, y_0) \end{pmatrix}.$$

A step size $h = 0.05$ is commonly used. The Euler increment $\vec{E}(\vec{v}_1)$ is given as scalar multiplication by h against an $\mathcal{R}^2$-vector which involves evaluation of f at the previous dot $\vec{v}_1$.

In summary, the **dots** for the graphic of $y(x)$ form a data set in the vector space $\mathcal{R}^2$. The dots are obtained by algorithm rules, which are easily expressed by vector addition ($+$) and scalar multiplication ($\cdot$). The 8 properties of the toolkit were used in a limited way.

Digital Photographs

A digital photo consists of many **pixels** arranged in a two dimensional array. Structure can be assigned to the photo by storing the pixel digital color data in a matrix A of size $n \times m$. Each entry of A is an integer which encodes the color information at a specific pixel location.

The set V of all $n \times m$ matrices is a vector space under the usual rules for matrix addition and scalar multiplication. Initially, V is just a storage system for photos. However, the algebraic toolkit for V is a convenient way to express operations on photos. We give one illustration: breaking a photo into RGB (Red, Green, Blue) separation photos, in order to make color separation transparencies.

One way to encode each entry of A is to define $a_{ij} = r_{ij} + g_{ij}x + b_{ij}x^2$ where x is some convenient base. The integers r_{ij}, g_{ij}, b_{ij} represent the amount of red, green and blue present in the pixel with data a_{ij}. Then $A = R + Gx + Bx^2$ where $R = [r_{ij}]$, $G = [g_{ij}]$, $B = [b_{ij}]$ are $n \times m$ matrices that represent the color separation photos. These monochromatic photos are superimposed as color transparencies on a standard overhead projector to duplicate the original photograph.

Printing machinery from many years ago employed separation negatives and multiple printing runs in primary ink colors to make book photos. The advent of digital printers and simpler inexpensive technologies has made the separation process nearly obsolete. To help the reader understand the historical events, we record the following quote from Sam Wang[8]:

> I encountered many difficulties when I first began making gum prints: it was not clear which paper to use; my exposing light (a sun lamp) was highly inadequate; plus a myriad of other problems. I was also using panchromatic film, making in–camera separations, holding RGB filters in front of the camera lens for three exposures onto 3 separate pieces of black and white film. I also made color separation negatives from color transparencies by enlarging in the darkroom. Both of these methods were not only tedious but often produced negatives very difficult to print — densities and contrasts that were hard to control and working in the dark with panchromatic film was definitely not fun. The fact that I got a few halfway decent prints is something of a small miracle, and represents hundreds of hours of frustrating work! Digital negatives by comparison greatly simplify the process. Nowadays (2004) I use color images from digital cameras as well as scans from slides, and the negatives print much more predictably.

Function Spaces

The default storage system used for applications involving ordinary or partial differential equations is a *function space*. The data item packages for differential equations are their solutions, which are *functions*, or in an applied context, a graphic defined on a certain graph window. They are **not** column vectors of numbers.

Functions and Column Vectors

An alternative view, adopted by researchers in numerical solutions of differential equations, is that a solution is a table of numbers, consisting of pairs of x and y values.

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[8]Sam Wang teaches photography and art with computer at Clemson University in South Carolina. His photography degree is from the University of Iowa (1966). **Reference**: *A Gallery of Tri-Color Prints*, by Sam Wang

These researchers might view a function as being a fixed vector. Their unique intuitive viewpoint is that a function is a **graph** and a graph is determined by so many **dots**, which are practically obtained by **sampling** the function $y(x)$ at a reasonably dense set of x-values. The approximation is

$$y \approx \begin{pmatrix} y(x_1) \\ y(x_2) \\ \vdots \\ y(x_n) \end{pmatrix}$$

where $x_1, \ldots, x_n$ are the **samples** and $y(x_1), \ldots, y(x_n)$ are the **sampled values** of function y.

The trouble with the approximation is that two different functions may need different sampling rates to properly represent their graphic. The result is that the two functions might need data storage systems of different dimensions, e.g., f needs its sampled values in $\mathcal{R}^{200}$ and g needs its sampled values in $\mathcal{R}^{400}$. The absence of a universal fixed vector storage system for sampled functions explains the appeal of a storage system like the set of all functions.

Infinitely Long Column Vectors. Novices suggest a way around the lack of a universal numerical data storage system for sampled functions: *develop a theory of column vectors with infinitely many components.* It may help you to think of any function f as an infinitely long column vector, with one entry $f(x)$ for each possible sample x, e.g.,

$$\vec{f} = \begin{pmatrix} \vdots \\ f(x) \\ \vdots \end{pmatrix} \qquad \text{level } x$$

It is not clear how to order or address the entries of such a column vector: at algebraic stages it hinders. Can computers store infinitely long column vectors? The easiest path through the algebra is to deal exactly with functions and function notation. Still, there is something attractive about the change from sampled approximations to a single column vector with infinite extent:

$$\vec{f} \approx \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix} \to \begin{pmatrix} \vdots \\ f(x) \\ \vdots \end{pmatrix} \qquad \text{level } x$$

The thinking behind the *level* x annotation is that x stands for one of the infinite possibilities for an invented sample. Alternatively, with a rich set of invented samples $x_1, \ldots, x_n$, value $f(x)$ equals approximately $f(x_j)$, where x is closest to some sample x_j.

The Vector Space V of all Functions on a Set E

The rules for function addition and scalar multiplication come from college algebra and pre-calculus backgrounds:

$$(f+g)(x) = f(x) + g(x), \quad (cf)(x) = c \cdot f(x).$$

These rules can be motivated and remembered by the level x notation of infinitely long column vectors:

$$c_1\vec{f} + c_2\vec{g} = c_1 \begin{pmatrix} \vdots \\ f(x) \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} \vdots \\ g(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ c_1 f(x) + c_2 g(x) \\ \vdots \end{pmatrix}$$

The rules define **addition** and **scalar multiplication** of functions. The closure law for a vector space holds. Routine, but tedious, justifications show that V, under the above rules for addition and scalar multiplication, has the required 8-property toolkit to make it a vector space:

Closure	The operations $f+g$ and kf are defined and result in a new function which is also in the set V of all functions on the set E.	
Addition	$f+g=g+f$	commutative
	$f+(g+h)=(f+g)+h$	associative
	The zero function 0 is defined and $0+f=f$	zero
	The function $-f$ is defined and $f+(-f)=0$	negative
Scalar multiply	$k(f+g)=kf+kg$	distributive I
	$(k_1+k_2)f=k_1f+k_2f$	distributive II
	$k_1(k_2f)=(k_1k_2)f$	distributive III
	$1f=f$	identity

Important subspaces of the vector space V of all functions appear in applied literature as the storage systems for solutions to differential equations and solutions of related models.

The Space $C(E)$

Let $E = \{x \;:\; a < x < b\}$ be an open interval on the real line. The set $C(E)$ is defined to be the subset S of the set V of all functions on E obtained by restricting the function to be continuous. Because sums and scalar multiples of continuous functions are continuous, then $S = C(E)$ is a subspace of V and a vector space in its own right. What has been said for an open interval E holds also for an open bounded set E.

The Space $C^1(E)$

The set $C^1(E)$ is the subset of the vector space $C(E)$ of all continuous functions on E obtained by restricting the function to be continuously differentiable. Because sums and scalar multiples of continuously differentiable functions are continuously differentiable, then $C^1(E)$ is a subspace of $C(E)$ and a vector space in its own right.

The Space $C^k(E)$

The set $C^k(E)$ is the subset of the vector space $C(E)$ of all continuous functions on E obtained by restricting the function to be k times continuously differentiable. Because sums and scalar multiples of k times continuously differentiable functions are k times continuously differentiable, then $C^k(E)$ is a subspace of $C(E)$ and a vector space in its own right.

Solution Space of a Differential Equation

The differential equation $y''-y=0$ has general solution $y=c_1e^x+c_2e^{-x}$, which means that the set S of all solutions of the differential equation consists of all possible linear combinations of the two functions e^x and e^{-x}. Briefly,

$$S = \mathbf{span}\left(e^x, e^{-x}\right).$$

The functions e^x, e^{-x} are in $C^2(E)$ for any interval E on the x-axis. Therefore, S is a subspace of $C^2(E)$ and a vector space in its own right.

More generally, every homogeneous linear differential equation, of any order, has a solution set S which is a vector space in its own right.

Other Vector Spaces

The number of different vector spaces used as data storage systems in scientific literature is finite, but growing with new discoveries. There is really no limit to the number of different settings possible, because creative individuals are able to invent new settings.

Here is an example of how creation begets new vector spaces. Consider the problem $y' = 2y + f(x)$ and the task of storing data for the plotting of an initial value problem with initial condition $y(x_0) = y_0$. The data set V suitable for plotting consists of column vectors

$$\vec{v} = \begin{pmatrix} x_0 \\ y_0 \\ f \end{pmatrix}.$$

A plot command takes such a data item, computes the solution

$$y(x) = y_0 e^{2x} + e^{2x} \int_0^x e^{-2t} f(t) dt$$

and then plots it in a window of fixed size with center at (x_0, y_0). The column vectors are not numerical vectors in $\mathcal{R}^3$, but some **hybrid** of vectors in $\mathcal{R}^2$ and the space of continuous functions $C(E)$ where E is the real line.

It is relatively easy to come up with definitions of vector addition and scalar multiplication on V. The closure law holds and the eight vector space properties can be routinely verified. Therefore, V is an abstract vector space, unlike any found in this text. We reiterate:

> An abstract vector space is a set V and two operations of $\boxed{+}$ and $\boxed{\cdot}$ such that the closure law holds and the eight algebraic vector space properties are satisfied.

The paycheck for having recognized a vector space setting in an application is clarity of exposition and economy of effort in details. Algebraic details in $\mathcal{R}^2$ can often be transferred unchanged to an abstract vector space setting, line for line, to obtain the details in the more abstract setting.

Independence and Dependence

The subject of *independence* applies to coordinate spaces $\mathcal{R}^n$, function spaces and in particular solution spaces of differential equations, digital photos, sequences of Fourier coefficients or Taylor coefficients, and general abstract vector spaces. Introduced here are definitions for low dimensions, the geometrical meaning of independence and basic algebraic tests for independence.

The motivation for the study of independence is the theory of general solutions, which are expressions representing *all possible solutions* of a linear problem. The subject of independence discovers *the shortest possible expression* for a general solution.

Definition 7 (Independence)
Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided each linear combination $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is represented by a **unique** set of constants $c_1, \ldots, c_k$.

Definition 8 (Dependence)
Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **dependent** provided they are not independent. This means that a linear combination $\vec{v} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$ can be represented in a second way as $\vec{v} = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$ where for at least one index j, $a_j \neq b_j$.

Independence means **unique representation of linear combinations** of $\vec{v}_1, \ldots, \vec{v}_k$, which is exactly the statement

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$$

implies the coefficients match:

$$\begin{cases} a_1 &= b_1 \\ a_2 &= b_2 \\ & \vdots \\ a_k &= b_k \end{cases}$$

Independence details normally use a briefer, more abstract equivalence, as in the theorem below. See Definition 9. The proof is delayed until page 386.

Theorem 21 ((Unique Representation of the Zero Vector))
Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are independent in vector space V if and only if the system of equations

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

has unique solution $c_1 = \cdots = c_k = 0$.

Independence of $1, x^2, x^4$ is decided by the following result, because it is known that powers $1, x, \ldots, x^4$ form an independent set. The proof is delayed until page 386.

Theorem 22 (Subsets of Independent Sets)
Any nonvoid subset of an independent set is also independent.

Subsets of dependent sets can be either independent or dependent.

Independence Test

To prove that vectors $\vec{v}_1, \ldots, \vec{v}_k$ are independent, form the system of equations

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Solve for the constants $c_1, \ldots, c_k$.

> **Independence** means all the constants $c_1, \ldots, c_k$ are zero.
>
> **Dependence** means that a **nonzero** solution $c_1, \ldots, c_k$ exists. This means $c_j \neq 0$ for at least one index j.

Geometric Independence and Dependence for Two Vectors

Two vectors $\vec{v}_1$, $\vec{v}_2$ in $\mathcal{R}^2$ are said to be **independent** provided neither is the zero vector and one is not a scalar multiple of the other. Graphically, this means $\vec{v}_1$ and $\vec{v}_2$ form the edges of a non-degenerate parallelogram.

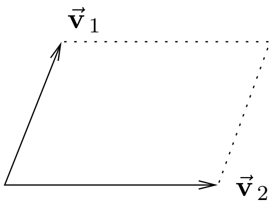

Figure 13. Independent vectors. Two nonzero nonparallel vectors $\vec{v}_1$, $\vec{v}_2$ form the edges of a parallelogram P. A vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ lies interior to P if the scaling constants satisfy $0 < c_1 < 1$, $0 < c_2 < 1$.

Algebraic Independence for Two Fixed Vectors

Given two vectors $\vec{v}_1$, $\vec{v}_2$, construct the system of equations in unknowns c_1, c_2

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}.$$

Solve the system for c_1, c_2. The two vectors are **independent** if and only if the system has the unique solution $c_1 = c_2 = 0$.

The test is equivalent to the statement that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ holds for one unique set of constants x_1, x_2. The details: if $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2$ and also $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2$, then subtraction of the two equations gives $(a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 = \vec{0}$. This is a relation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ with $c_1 = a_1 - b_1$, $c_2 = a_2 - b_2$. Independence means $c_1 = c_2 = 0$, or equivalently, $a_1 = b_1$, $a_2 = b_2$. The details are complete.

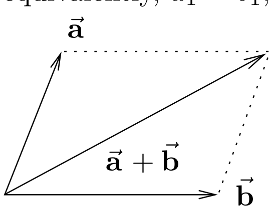

Figure 14. The parallelogram rule. Two nonzero vectors $\vec{a}$, $\vec{b}$ are added by the parallelogram rule: vector $\vec{a} + \vec{b}$ has tail matching the joined tails of $\vec{a}$, $\vec{b}$ and head at the corner of the completed parallelogram.

Why does the test work? Vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ is formed by the parallelogram rule, Figure 14, by adding the scaled vectors $\vec{a} = c_1\vec{v}_1$, $\vec{b} = c_2\vec{v}_2$. The zero vector $\vec{v} = \vec{0}$ can be obtained from nonzero nonparallel vectors $\vec{v}_1$, $\vec{v}_2$ only if the scaling factors c_1, c_2 are both zero.

Geometric Dependence of Two Fixed Vectors

Define vectors $\vec{v}_1$, $\vec{v}_2$ in $\mathcal{R}^2$ to be **dependent** provided they are **not independent**. This means one of $\vec{v}_1$, $\vec{v}_2$ is the zero vector or else $\vec{v}_1$ and $\vec{v}_2$ lie along the same line: *the two vectors cannot form a parallelogram.* Algebraic detection of dependence is by failure of the independence test: after solving the system $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, one of the two constants c_1, c_2 is nonzero.

Fixed Vector Illustration

Two column vectors are tested for independence by forming the system of equations $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, e.g,

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This vector equation can be written as a homogeneous system $A\vec{c} = \vec{0}$, defined by

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The system $A\vec{c} = \vec{0}$ can be solved for $\vec{c}$ by **rref** methods. Because **rref**$(A) = I$, then $c_1 = c_2 = 0$, which verifies independence of the two vectors.

If A is square and **rref**$(A) = I$, then A^{-1} exists. The equation $A\vec{c} = \vec{0}$ can be solved by multiplication of both sides by A^{-1}. Then the unique solution is $\vec{c} = \vec{0}$, which means $c_1 = c_2 = 0$. Inverse theory says $A^{-1} = \mathbf{adj}(A)/\det(A)$ exists precisely when $\det(A) \neq 0$, therefore independence is verified independently of **rref** methods by the 2×2 determinant computation $\det(A) = -3 \neq 0$.

Remarks about $\det(A)$ apply to independence testing for any two vectors, but only in case the system of equations $A\vec{c} = \vec{0}$ is square. For instance, in $\mathcal{R}^3$, the homogeneous system

$$c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has vector-matrix form $A\vec{c} = \vec{0}$ with 3×2 matrix A. There is **no chance to use determinants** directly. Left undiscussed are clever determinant methods for $m \times n$ systems, because we rely on **rref** methods for such systems.

Geometric Independence and Dependence for Three Vectors

Three vectors in $\mathcal{R}^3$ are said to be independent provided none of them are the zero vector and they form the edges of a non-degenerate parallelepiped of positive volume. Such vectors are called a **triad**. In the special case of all pairs orthogonal (the vectors are 90° apart) they are called an **orthogonal triad**.

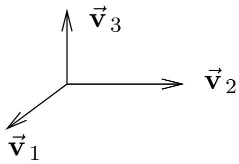

Figure 15. Independence of three vectors. Vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ form the edges of a parallelepiped. A vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ is located interior to the parallelepiped, provided satisfying $0 < c_1, c_2, c_3 < 1$.

Algebraic Independence Test for Three Vectors

Given vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, construct the vector equation in unknowns c_1, c_2, c_3

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Solve the system for c_1, c_2, c_3. The vectors are **independent** if and only if the system has unique solution $c_1 = c_2 = c_3 = 0$.

Why does the test work? The vector $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ is formed by two applications of the parallelogram rule: first add the scaled vectors $c_1\vec{v}_1$, $c_2\vec{v}_2$ and secondly add the scaled vector $c_3\vec{v}_3$ to the resultant. The zero vector $\vec{v} = \vec{0}$ can be obtained from a vector triad $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ only if the scaling factors c_1, c_2, c_3 are all zero.

Geometric Dependence of Three Vectors

Given vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, they are **dependent** if and only if they are **not independent**. The three subcases that occur can be analyzed geometrically by the theorem proved earlier:

> A nonvoid subset of an independent set is independent.

The three cases:

1. There is a dependent subset of one vector. Then one of them is the zero vector.

2. There is a dependent subset of two vectors. Then two of them lie along the same line.

2. There is a dependent subset of three vectors. Then one of them is in the plane of the other two.

In summary, three dependent vectors in $\mathcal{R}^3$ cannot be the edges of a parallelepiped. Algebraic detection of dependence is by failure of the independence test: after solving the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, one of the three constants c_1, c_2, c_3 is nonzero[9].

[9] In practical terms, there is at least one free variable, or equivalently, appearing in the solution formula is at least one invented symbol t_1, t_2, $\ldots$.

Independence in an Abstract Vector Space

Linear algebra literature might assume a **different basic definition** of independence, which is purely algebraic:

Definition 9 (Independence in an Abstract Vector Space)
Let $\vec{v}_1, \ldots, \vec{v}_k$ be a finite set of vectors in an abstract vector space V. The set is called **independent** if and only if the system of equations in unknowns $c_1, \ldots, c_k$

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

has unique solution $c_1 = \cdots = c_k = 0$.

The set of vectors is called **dependent** if and only if it is not independent. This means that the system of equations in variables $c_1, \ldots, c_k$ has a solution with at least one variable c_j nonzero.

Independence defined in this abstract manner means that each linear combination $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is represented by a unique set of constants $c_1, \ldots, c_k$, as in Definition 7. See Theorem 23 and its consequences in Theorem 24 and Theorem 25. Proofs are in the exercises, page 389.

It is not obvious how to solve for $c_1, \ldots, c_k$ in the algebraic independence test, when the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are not fixed vectors. If V is a set of functions, then the methods from linear algebraic equations do not directly apply. This algebraic problem causes us to develop special tools just for functions, called the **sampling test** and **Wronskian test**. Examples appear later, which illustrate how to apply these two important independence tests for functions.

Theorem 23 (Unique Representation)
Let $\vec{v}_1, \ldots, \vec{v}_k$ be independent vectors in an abstract vector space V. If scalars $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ satisfy the relation

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k$$

then the coefficients must match:

$$\begin{cases} a_1 & = & b_1, \\ a_2 & = & b_2, \\ \vdots & & \\ a_k & = & b_k. \end{cases}$$

Theorem 24 (Independence of Two Vectors)
Two vectors in an abstract vector space V are independent if and only if neither is the zero vector and each is not a constant multiple of the other.

Theorem 25 (Zero Vector)
An independent set in an abstract vector space V cannot contain the zero vector. Moreover, an independent set cannot contain a vector which is a linear combination of the others.

Independence and Dependence Tests for Fixed Vectors

Recorded here are a number of useful algebraic tests to determine independence or dependence of a finite list of fixed vectors.

Rank Test

In the vector space $\mathcal{R}^n$, the key to detection of independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. The test is justified from the formula $\mathbf{nullity}(A) + \mathbf{rank}(A) = k$, where k is the column dimension of A.

Theorem 26 (Rank-Nullity Test for Three Vectors)
Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be 3 column vectors in $\mathcal{R}^n$ and let their $n \times 3$ augmented matrix be

$$A = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle.$$

The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent if $\mathbf{rank}(A) = 3$ and dependent if $\mathbf{rank}(A) < 3$. The conditions are equivalent to $\mathbf{nullity}(A) = 0$ and $\mathbf{nullity}(A) > 0$, respectively.

Theorem 27 (Rank-Nullity Test)
Let $\vec{v}_1, \ldots, \vec{v}_k$ be k column vectors in $\mathcal{R}^n$ and let A be their $n \times k$ augmented matrix. The vectors are independent if $\mathbf{rank}(A) = k$ and dependent if $\mathbf{rank}(A) < k$. The conditions are equivalent to $\mathbf{nullity}(A) = 0$ and $\mathbf{nullity}(A) > 0$, respectively.

The proofs are delayed until page 387.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $A\vec{c} = \vec{0}$ and A is square, then $\det(A) = 0$ detects dependence, and $\det(A) \neq 0$ detects independence. The reasoning is based upon the formula $A^{-1} = \mathbf{adj}(A)/\det(A)$, valid exactly when $\det(A) \neq 0$.

Theorem 28 (Determinant Test)
Let $\vec{v}_1, \ldots, \vec{v}_n$ be n column vectors in $\mathcal{R}^n$ and let A be the $n \times n$ augmented matrix of these vectors. The vectors are independent if $\det(A) \neq 0$ and dependent if $\det(A) = 0$.

The proof is delayed until page 387.

Orthogonal Vector Test

In some applications the vectors being tested are known to satisfy **orthogonality conditions**. For three vectors, these conditions are written

$$(1) \qquad \begin{array}{lll} \vec{v}_1 \cdot \vec{v}_1 > 0, & \vec{v}_2 \cdot \vec{v}_2 > 0, & \vec{v}_3 \cdot \vec{v}_3 > 0, \\ \vec{v}_1 \cdot \vec{v}_2 = 0, & \vec{v}_2 \cdot \vec{v}_3 = 0, & \vec{v}_3 \cdot \vec{v}_1 = 0. \end{array}$$

The equations mean that the vectors are nonzero and pairwise 90° apart. The set of vectors is said to be **pairwise orthogonal**, or briefly, **orthogonal**. For a list of k vectors, the orthogonality conditions are written

$$(2) \qquad \vec{v}_i \cdot \vec{v}_i > 0, \quad \vec{v}_i \cdot \vec{v}_j = 0, \quad 1 \le i, j \le k, \quad i \ne j.$$

Theorem 29 (Orthogonal Vector Test)
A set of nonzero pairwise orthogonal vectors $\vec{v}_1, \ldots, \vec{v}_k$ is linearly independent.

The proof is delayed until page 387.

Independence Tests for Functions

Recorded here are a number of useful algebraic tests to determine independence of a finite list of functions. Neither test is an equivalence. A test applies to determine independence, but dependence is left undetermined. No results here imply that a list of functions is dependent.

Sampling Test for Functions

Let f_1, f_2, f_3 be three functions defined on a domain D. Let V be the vector space of all functions $\vec{f}$ on D with the usual scalar multiplication and addition rules learned in college algebra.[10] Addressed here is the question of how to test independence and dependence of $\vec{f}_1$, $\vec{f}_2$, $\vec{f}_3$ in V. The vector relation

$$c_1\vec{f}_1 + c_2\vec{f}_2 + c_3\vec{f}_3 = \vec{0}$$

means

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0, \quad x \text{ in } D.$$

An idea how to solve for c_1, c_2, c_3 arises by **sampling**, which means 3 relations are obtained by **inventing** 3 values for x, say x_1, x_2, x_3. The equations arising are

$$\begin{array}{ccccccc} c_1 f_1(x_1) & + & c_2 f_2(x_1) & + & c_3 f_3(x_1) & = & 0, \\ c_1 f_1(x_2) & + & c_2 f_2(x_2) & + & c_3 f_3(x_2) & = & 0, \\ c_1 f_1(x_3) & + & c_2 f_2(x_3) & + & c_3 f_3(x_3) & = & 0. \end{array}$$

[10]Symbol $\vec{f}$ is the vector package for function f. Symbol $f(x)$ is a number, a function value. Symbol f is a graph, equivalently the domain D plus equation $y = f(x)$.

This system of 3 equations in 3 unknowns can be written in matrix form $A\vec{c} = \vec{0}$, where the coefficient matrix A and vector $\vec{c}$ of unknowns c_1, c_2, c_3 are defined by

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

The matrix A is called the **sampling matrix** for f_1, f_2, f_3 with **samples** x_1, x_2, x_3.

The system $A\vec{c} = \vec{0}$ has unique solution $\vec{c} = \vec{0}$, proving $\vec{f_1}$, $\vec{f_2}$, $\vec{f_3}$ independent, provided $\det(A) \neq 0$.

All of what has been said here for three functions applies to k functions $f_1, \ldots, f_k$, in which case k samples $x_1, \ldots, x_k$ are invented. The sampling matrix A and vector $\vec{c}$ of variables are then

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_k(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_k(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_k) & f_2(x_k) & \cdots & f_k(x_k) \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Theorem 30 (Sampling Test for Functions)
The functions $f_1, \ldots, f_k$ are linearly independent on an x-set D provided there is a sampling matrix A constructed from invented samples $x_1, \ldots, x_k$ in D such that $\det(A) \neq 0$.

It is **false** that independence of the functions implies $\det(A) \neq 0$. The relation $\det(A) \neq 0$ depends on the invented samples.

Wronskian Test for Functions

The test will be explained first for two functions f_1, f_2. Independence of f_1, f_2, as in the sampling test, is decided by solving for constants c_1, c_2 in the equation

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x.$$

J. M. Wronski suggested to solve for the constants by differentiation of this equation, obtaining a pair of equations

$$\begin{array}{rcrcl} c_1 f_1(x) & + & c_2 f_2(x) & = & 0, \\ c_1 f_1'(x) & + & c_2 f_2'(x) & = & 0, \quad \text{for all } x. \end{array}$$

This is a system of equations $A\vec{c} = \vec{0}$ with coefficient matrix A and variable list vector $\vec{c}$ given by

$$A = \begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The **Wronskian Test** is simply $\det(A) \neq 0$ implies $\vec{c} = \vec{0}$, similar to the sampling test:

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \neq 0 \quad \text{for some } x \text{ implies } f_1,\ f_2 \text{ independent.}$$

Interesting about Wronski's idea is that it requires the invention of just one sample x such that the determinant is non-vanishing, in order to establish independence of the two functions.

Wronskian Test for n Functions. Given functions $f_1, \ldots, f_n$ each differentiable $n-1$ times on an interval $a < x < b$, the **Wronskian determinant** [11] is defined by the relation

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem 31 (Wronskian Test)
Let functions $f_1, \ldots, f_n$ be differentiable $n-1$ times on interval $a < x < b$. Then $W(f_1, \ldots, f_n)(x_0) \neq 0$ for some x_0 in (a, b) implies $f_1, \ldots, f_n$ are independent functions in the vector space V of all functions on (a, b).

The proof parallels the one for the sampling test, delayed to page 388.

Euler Solution Atom Test

The test originates in linear differential equations. It applies in a variety of situations outside that scope, providing basic intuition about independence of functions. It can be proved from the Wronskian test, but we don't try to do that. See Example 20, page 384.

Definition 10 (Euler Solution Atom)
A **base atom** is one of $1, \cos(bx), \sin(bx)$ or $e^{ax}, e^{ax}\cos(bx), e^{ax}\sin(bx)$, where $b > 0$ and $a \neq 0$. An **Euler solution atom** is a power x^n times a base atom, n a non-negative integer (0,1,2,...).

Theorem 32 (Independence of Euler Solution Atoms)
A finite list of distinct Euler solution atoms is independent.

[11]Named after mathematician J. M. Wronski (1776-1853). Born Józef Maria Hoëné in Poland, he resided his final 40 years in France using the name Wronski.

Application: Vandermonde Determinant

Choosing the functions in the *sampling test* to be 1, x, x^2 with invented samples x_1, x_2, x_3 gives the sampling matrix

$$V(x_1, x_2, x_3) = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}.$$

The sampling matrix is called a **Vandermonde matrix**. Using the polynomial basis $f_1(x) = 1$, $f_2(x) = x$, ..., $f_k(x) = x^{k-1}$ and invented samples x_1, ..., x_k gives the $k \times k$ Vandermonde matrix

$$V(x_1, \ldots, x_k) = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{k-1} \\ 1 & x_2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_k & \cdots & x_k^{k-1} \end{pmatrix}.$$

The most often used Vandermonde determinant identities are

$$\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} = b - a,$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-b)(c-a)(b-a),$$

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a).$$

Theorem 33 (Vandermonde Determinant Identity)
The Vandermonde matrix has a nonzero determinant for distinct samples, because of the identity

$$\det(V(x_1, \ldots, x_k)) = \prod_{i<j} (x_j - x_i).$$

The technically demanding mathematical induction proof is delayed until page 388.

Examples

14 **Example (Vector General Solution)** Find the vector general solution $\vec{\mathbf{u}}$ of $A\vec{\mathbf{u}} = \vec{\mathbf{0}}$, given matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: The solution divides into two distinct sections: **1** and **2**.

1: Find the scalar general solution of the system $A\vec{x} = \vec{0}$.

The toolkit: combination, swap and multiply. Then we use the last frame algorithm. The usual shortcut applies to compute $\mathbf{rref}(A)$. We skip the augmented matrix $\langle A|\vec{0}\rangle$, knowing that the last column of zeros is unchanged by the toolkit. The details:

$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	First frame.
$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	`combo(1,2,-2)`.
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	`combo(2,1,-2)`. Last frame, this is $\mathbf{rref}(A)$.
$\left\lvert\begin{array}{rcl} x_1 &=& 0, \\ x_2 &=& 0, \\ 0 &=& 0. \end{array}\right\rvert$	Translate to scalar equations.
$\left\lvert\begin{array}{rcl} x_1 &=& 0, \\ x_2 &=& 0, \\ x_3 &=& t_1. \end{array}\right\rvert$	Scalar general solution, obtained from the last frame algorithm: x_1, x_2=lead, x_3=free.

2: Find the vector general solution of the system $A\vec{x} = \vec{0}$.

The plan is to use the answer from **1** and partial differentiation to display the vector general solution $\vec{x}$.

$\left\lvert\begin{array}{rcl} x_1 &=& 0, \\ x_2 &=& 0, \\ x_3 &=& t_1. \end{array}\right\rvert$	Scalar general solution, from **1**.
$\partial_{t_1}\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	The **special solution** is the partial on symbol t_1. Only one, because there is only one invented symbol.
$\vec{x} = t_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	The vector general solution. It is the sum of terms, an invented symbol times the corresponding **special solution** (partial on that symbol). See Example 17 for more details.

15 Example (Independence) Assume $\vec{v}_1, \vec{v}_2$ are independent vectors in abstract vector space V. Display the details which verify the independence of the vectors $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$.

Solution: The independence test of Theorem 21 will be applied. We must somehow solve for c_1, c_2 in the equation

$$c_1 (\vec{v}_1 + 3\vec{v}_2) + c_2 (\vec{v}_1 - 2\vec{v}_2) = \vec{0}.$$

The plan is to re-write this equation in terms of $\vec{v}_1, \vec{v}_2$, then use independence of vectors $\vec{v}_1, \vec{v}_2$ to obtain scalar equations for c_1, c_2. The equation re-arrangement:

$$(c_1 + c_2)\,\vec{v}_1 + (3c_1 - 2c_2)\,\vec{v}_2 = \vec{0}.$$

Independence according to Theorem 21 means that any relation $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$ implies scalar equations $a = 0$, $b = 0$. The re-arranged equation has $a = c_1 + c_2$, $b = 3c_1 - 2c_2$. Therefore, independence strips away the vectors from the re-arranged equation, leaving a system of scalar equations in symbols c_1, c_2:

$$\begin{array}{rcrcll} c_1 & + & c_2 & = & 0, & \text{The equation } a = 0, \\ 3c_1 & - & 2c_2 & = & = 0, & \text{The equation } b = 0. \end{array}$$

These equations have only the zero solution $c_1 = c_2 = 0$, because the coefficient matrix $\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$ is invertible (determinant nonzero). This completes the proof that vectors $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$ are independent.

16 Example (Span) Let $\vec{v}_1, \vec{v}_2$ be two vectors in an abstract vector space V. Define two subspaces

$$S_1 = \mathbf{span}(\vec{v}_1, \vec{v}_2), \quad S_2 = \mathbf{span}(\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2).$$

(a) Display the technical details which show that the two subspaces are equal: $S_1 = S_2$.

(b) Use the result of (a) to prove that independence of $\vec{v}_1, \vec{v}_2$ implies independence of $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$.

Solution:

Details for (a). Sets S_1, S_2 are known to be subspaces of V. To show $S_1 = S_2$, we will show each set is a subset of the other, that is, $S_2 \subset S_1$ and $S_1 \subset S_2$.

Show $S_2 \subset S_1$. By definition of **span**, both vectors $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$ belong to the set S_1. Therefore, the span of these two vectors is also in subspace S_1, hence $S_2 \subset S_1$.

Show $S_1 \subset S_2$. We will write $\vec{v}_1$ as a linear combination of $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$ in two steps. Then $\vec{v}_1$ belongs to S_2.

Step 1.Eliminate $\vec{v}_2$ with a combination.

$$5\vec{v}_1 = 2(\vec{v}_1 + 3\vec{v}_2) + 3(\vec{v}_1 - 2\vec{v}_2).$$

Step 2. Divide by 5.

$$\vec{v}_1 = \frac{2}{5}(\vec{v}_1 + 3\vec{v}_2) + \frac{3}{5}(\vec{v}_1 - 2\vec{v}_2).$$

Similarly, $\vec{v}_2$ belongs to S_2. Therefore, the span of $\vec{v}_1, \vec{v}_2$ belongs to S_2, or $S_1 \subset S_2$, as claimed.

Details for (b). Independence of $\vec{v}_1, \vec{v}_2$ implies $\dim(S_1) = 2$. Therefore, $\dim(S_2) = 2$. If $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$ fail to be independent, then dependency implies $\dim(S_2) \le 1$. We conclude from $\dim(S_2) = 2$ that $\vec{v}_1 + 3\vec{v}_2, \vec{v}_1 - 2\vec{v}_2$ are independent.

17 Example (Independence, Span and Basis)
A 5×5 linear system $A\vec{x} = \vec{0}$ has scalar general solution

$$\begin{aligned} x_1 &= t_1 + 2t_2, \\ x_2 &= t_1, \\ x_3 &= t_2, \\ x_4 &= 4t_2 + t_3, \\ x_5 &= t_3. \end{aligned}$$

Find a basis for the solution space.

Solution: The answer is the set of **special solutions** obtained by taking partial derivatives on the symbols t_1, t_2, t_3. Differentiation details are below.

$$\vec{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{X}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{X}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Span. The vector general solution is expressed as the sum $\vec{x} = t_1\vec{X}_1 + t_2\vec{X}_2 + t_3\vec{X}_3$, which implies that the solution space is $\mathbf{span}(\vec{X}_1, \vec{X}_2, \vec{X}_3)$.

Independence We repeat the details for independence of the three special solutions $\vec{X}_1, \vec{X}_2, \vec{X}_3$, a fact proved earlier in the text. The independence test in Theorem 21 is applied, which dictates solving for c_1, c_2, c_3 in the equation $c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3 = \vec{0}$. The vector general solution with $t_1 = c_1, t_2 = c_2, t_3 = c_3$ says that $\vec{x} = \vec{0}$, which in scalar form means $x_1 = x_2 = x_3 = x_4 = x_5 = 0$. The scalar general solution, specialized to the free variable equations

$$\begin{aligned} x_2 &= t_1, \\ x_3 &= t_2, \\ x_5 &= t_3, \end{aligned}$$

implies that $t_1 = t_2 = t_3 = 0$ (substitute $x_2 = x_3 = x_5 = 0$). Then $t_1 = t_2 = t_3 = 0$ implies $c_1 = c_2 = c_3 = 0$, proving independence of $\vec{X}_1, \vec{X}_2, \vec{X}_3$. A similar argument proves the general result:

Lemma 1 The **Special Solutions** for a linear homogeneous system $A\vec{x} = \vec{0}$ are linearly independent.

Special Solution Details. The plan is to take the partial derivative of the scalar general solution on symbol t_1. This creates special solution $\vec{X}_1$. The others are found the same way, by partials on t_2, t_3. For symbol t_1:

$$\partial_{t_1}\vec{x} = \begin{pmatrix} \partial_{t_1}x_1 \\ \partial_{t_1}x_2 \\ \partial_{t_1}x_3 \\ \partial_{t_1}x_4 \\ \partial_{t_1}x_5 \end{pmatrix} = \begin{pmatrix} \partial_{t_1}(t_1 + 2t_2) \\ \partial_{t_1}(t_1) \\ \partial_{t_1}(t_2) \\ \partial_{t_1}(4t_2 + t_3) \\ \partial_{t_1}(t_3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

18 Example (Rank Test and Determinant Test) Apply both the rank test and the determinant test to decide independence or dependence of the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

Solution: Answer: The vectors are dependent.

Details for the Rank Test. The plan is to form the augmented matrix A of the four vectors and then compute its rank. If the rank is 4, then they are independent, otherwise they are dependent, by the rank test.

$$\begin{aligned} A &= \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \vec{v}_4 \rangle \\ &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}. \end{aligned}$$

How to determine that the rank is not 4? We use the fact that the rank of A equals the rank of A^T. Equivalently, the row rank equals the column rank. Then a row of zeros implies a dependent set of rows, which implies the row rank is 3 or less (the rank is not 3). Also, columns one and two of A are identical, impossible for independent columns, therefore the column rank is not 4.

Details for the Determinant Test. The test uses the matrix A defined above. The question of independence reduces to testing $|A|$ nonzero. If nonzero, then the columns of A are independent, which implies the four given vectors are independent. Otherwise, $|A| = 0$ which implies the columns of A are dependent, in turn the given four vectors are dependent.

All of this depends upon A being square: there is no determinant theory for non-square matrices.

Compute $|A| = 0$, because A has a row of zeros. Also, $|A| = 0$ because A has duplicate columns. Therefore, the columns of A are dependent, which translates to the given four vectors being dependent.

19 Example (Sampling Test and Wronskian Test) Let $V = C(-\infty, \infty)$ and define vectors $\vec{v}_1 = x^2$, $\vec{v}_2 = x^{7/3}$, $\vec{v}_3 = x^5$.[12] Apply the sampling test and the Wronskian test to establish independence of the three vectors in V.

Solution: We mention that the vectors are not Euler solution atoms, therefore there is no shortcut to decide on independence. However, intuition obtained from the Euler solution atom test suggests that the three vectors are independent.

Also important to note is that the vectors are not fixed vectors (column vectors), therefore the rank test and determinant test cannot apply.

[12]To write $\vec{v}_1 = x^2$ defines vector package $\vec{v}_1$ in V with domain $(-\infty, \infty)$ and equation $y = x^2$.

Sampling Test Details. A bad sample choice is $x = 0$, because it will produce a row of zeros, hence a zero determinant, leading to no test. We choose samples $x = 1, 2, 3$ for lack of insight, and then see if it works. The **sample matrix** is obtained by replacing $x = 1, 2, 3$ respectively into the row vector $\left(x^2, x^{7/3}, x^5 \right)$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & (\sqrt[3]{2})^7 & 32 \\ 9 & (\sqrt[3]{3})^7 & 243 \end{pmatrix}.$$

Because $|A| \approx 132$ is nonzero, then the given vectors are independent by the sampling test.

Wronskian Test Details. We'll choose the sample x after finding the Wronskian matrix $W(x)$ for all x. Start with row vector $\left(x^2, x^{7/3}, x^5 \right)$ and differentiate twice to compute the rows of the Wronskian matrix:

$$W(x) = \begin{pmatrix} x^2 & x^{7/3} & x^5 \\ 2x & \frac{7}{3}x^{4/3} & 5x^4 \\ 2 & \frac{28}{9}x^{1/3} & 20x^3 \end{pmatrix}.$$

The sample $x = 0$ won't work, because $|W(0)|$ has a row of zeros. We choose $x = 1$, then

$$W(1) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & \frac{7}{3} & 5 \\ 2 & \frac{28}{9} & 20 \end{pmatrix}.$$

The determinant $|W(1)| = 8/3$ is nonzero, which implies the three functions are independent by the Wronskian test.

20 Example (Solution Space of a Differential Equation) A fifth order linear differential equation has general solution

$$y(x) = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 e^{2x}.$$

Write the solution space S in vector space $C^5(-\infty, \infty)$ as the span of basis vectors.

Solution: The answer is

$$S = \textbf{span}\left(1, x, e^x, e^{-x}, e^{2x}\right).$$

Details. A general solution is an expression for all solutions (no solutions skipped) in terms of arbitrary constants, in this case, the constants c_1 to c_5. We think of the constants as the invented symbols $t_1, t_2, \ldots$ in a matrix equation general solution. Then the expected basis vectors should be the partial derivatives on the symbols:

$$\begin{aligned} \partial_{c_1} y(x) &= 1, \\ \partial_{c_2} y(x) &= x, \\ \partial_{c_3} y(x) &= e^x, \\ \partial_{c_4} y(x) &= e^{-x}, \\ \partial_{c_5} y(x) &= e^{2x}. \end{aligned}$$

The five vectors so obtained already span the space S. All that remains is to prove they are independent. The easiest method to apply in this case is the Wronskian test.

Independence Details. Let $W(x)$ be the Wronskian of the five solutions above. Then row one is the list $1, x, e^x, e^{-x}, e^{2x}$ and the other four rows are successive derivatives of the first row.

$$W(x) = \begin{vmatrix} 1 & x & e^x & e^{-x} & e^{2x} \\ 0 & 1 & e^x & -e^{-x} & 2e^{2x} \\ 0 & 0 & e^x & e^{-x} & 4e^{2x} \\ 0 & 0 & e^x & -e^{-x} & 8e^{2x} \\ 0 & 0 & e^x & e^{-x} & 16e^{2x} \end{vmatrix}.$$

The cofactor rule applied twice in succession to column 1 gives

$$W(x) = \begin{vmatrix} e^x & e^{-x} & 4e^{2x} \\ e^x & -e^{-x} & 8e^{2x} \\ e^x & e^{-x} & 16e^{2x} \end{vmatrix}.$$

We choose sample $x = 0$ to simplify the work:

$$W(0) = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 8 \\ 1 & 1 & 16 \end{vmatrix} = -24.$$

Then the determinant $|W(0)| = -24$ is nonzero, which implies independence of the functions in row one of $W(x)$, by the Wronskian test.

A Faster Independence Test. Generally, we skip the Wronskian test and apply the Euler solution atom test, in Theorem 32, which dispenses with independence in a few seconds.[13]

The details of the Euler solution atom test are brief: we check that the list $1, x, e^x, e^{-x}, e^{2x}$ is a finite set of distinct Euler solution atoms, then apply the test to conclude that the set is independent.

21 Example (Extracting a Basis from a List) In the vector space V of all polynomials, consider the subspace $S = \mathbf{span}(x+1, 2x-1, 3x+4, x^2)$. Find a basis for S selected from the list $x+1, 2x-1, 3x+4, x^2$.

Solution: The answer: $x+1, 2x-1, x^2$.

The vectors $x+1, 2x-1$ are independent, because one is not a scalar multiple of the other (they are lines with slopes $1, 2$).

The list $x+1, 2x-1, 3x+4$ of three vectors is dependent. In detail, we first will show $\mathbf{span}(x+1, 2x-1) = \mathbf{span}(1, x)$, using these two stages:

$\boxed{1}$: $3x = (x+1) + (2x-1)$

$\boxed{2}$: $-3 = -2(x+1) + (2x-1)$

[13] The proof of the Euler solution atom test, not supplied in this textbook, involves determinant evaluations similar to this example. Essential to the proof is the fact that subsets of independent sets are independent.

Divide by 3 and -3 to show $\mathbf{span}(x+1, 2x-1) = \mathbf{span}(1, x)$. Then $3x+4$ is in $\mathbf{span}(x+1, 2x-1)$. We skip $3x+4$ and go on to add x^2 to the list. Vector x^2 is not in $\mathbf{span}(x+1, 2x-1) = \mathbf{span}(1, x)$, because the Euler solution atoms $1, x, x^2$ are independent. The final independent set is $x+1, 2x-1, x^2$, and this is a basis for S.

Details and Proofs

Proof of Theorem 21, Unique Representation of the Zero Vector: The proof will be given for the characteristic case $k = 3$, because details for general k can be written from this proof, by minor editing of the text.

Assume vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent and $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Then $a_1\vec{v}_1 + x_2\vec{v}_2 + a_3\vec{v}_3 = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3$ where we define $a_1 = c_1$, $a_2 = c_2$, $a_3 = c_3$ and $b_1 = b_2 = b_3 = 0$. By independence, the coefficients match. By the definition of the symbols, this implies the equations $c_1 = a_1 = b_1 = 0$, $c_2 = a_2 = b_2 = 0$, $c_3 = a_3 = b_3 = 0$. Then $c_1 = c_2 = c_3 = 0$.

Conversely, assume $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ implies $c_1 = c_2 = c_3 = 0$. If

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3,$$

then subtract the right side from the left to obtain

$$(a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + (a_3 - b_3)\vec{v}_3 = \vec{0}.$$

This equation is equivalent to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

where the symbols c_1, c_2, c_3 are defined by $c_1 = a_1 - b_1$, $c_2 = a_2 - b_2$, $c_3 = a_3 - b_3$. The theorem's condition implies that $c_1 = c_2 = c_3 = 0$, which in turn implies $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$. The proof is complete.

Proof of Theorem 22, Subsets of Independent Sets: The idea will be communicated for a set of three independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Let the subset to be tested consist of the two vectors $\vec{v}_1, \vec{v}_2$. We form the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

and solve for the constants c_1, c_2. If $c_1 = c_2 = 0$ is the only solution, then $\vec{v}_1, \vec{v}_2$ is a an independent set.

Define $c_3 = 0$. Because $c_3\vec{v}_3 = \vec{0}$, the term $c_3\vec{v}_3$ can be added into the previous vector equation to obtain the new vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Independence of the three vectors implies $c_1 = c_2 = c_3 = 0$, which in turn implies $c_1 = c_2 = 0$, completing the proof that $\vec{v}_1, \vec{v}_2$ are independent.

The proof for an arbitrary independent set $\vec{v}_1, \ldots, \vec{v}_k$ is similar. By renumbering, we can assume the subset to be tested for independence is $\vec{v}_1, \ldots, \vec{v}_m$ for some index $m \leq k$. The proof amounts to adapting the proof for $k = 3$ and $m = 2$, given above. The details are left to the reader.

Because a single nonzero vector is an independent subset of any list of vectors, then a subset of a dependent set can be independent. If the subset of the dependent set is the whole set, then the subset is dependent. In conclusion, subsets of dependent sets can be either independent or dependent.

Proof of Theorem 27, Rank-Nullity Test: The proof will be given for $k = 3$, because a small change in the text of this proof is a proof for general k.

Suppose $\mathbf{rank}(A) = 3$. Then there are 3 leading ones in $\mathbf{rref}(A)$ and zero free variables. Therefore, $A\vec{c} = \vec{0}$ has unique solution $\vec{c} = \vec{0}$.

The independence of the 3 vectors is decided by solving the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

for the constants c_1, c_2, c_3. The vector equation says that a linear combination of the columns of matrix A is the zero vector, or equivalently, $A\vec{c} = \vec{0}$. Therefore, $\mathbf{rank}(A) = 3$ implies $\vec{c} = \vec{0}$, or equivalently, $c_1 = c_2 = c_3 = 0$. This implies that the 3 vectors are linearly independent.

If $\mathbf{rank}(A) < 3$, then there exists at least one free variable. Then the equation $A\vec{c} = \vec{0}$ has at least one nonzero solution $\vec{c}$. This implies that the vectors are dependent.

The proof is complete.

Proof of Theorem 28, Determinant Test: The proof details will be done for $n = 3$, because minor edits to this text will give the details for general n.

Algebraic independence of vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ requires solving the system of equations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

for constants c_1, c_2, c_3. The left side of the equation is a linear combination of the columns of the augmented matrix $A = \langle \vec{v}_1|\vec{v}_2|\vec{v}_3\rangle$, and therefore the system can be represented as the matrix equation $A\vec{c} = \vec{0}$. If $\det(A) \neq 0$, then A^{-1} exists. Multiply the equation $A\vec{c} = \vec{0}$ by the inverse matrix to give

$$\begin{aligned} A\vec{c} &= \vec{0} \\ A^{-1}A\vec{c} &= A^{-1}\vec{0} \\ I\vec{c} &= A^{-1}\vec{0} \\ \vec{c} &= \vec{0}. \end{aligned}$$

Then $\vec{c} = \vec{0}$, or equivalently, $c_1 = c_2 = c_3 = 0$. The vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent.

Conversely, if the vectors are independent, then the system $A\vec{c} = \vec{0}$ has a unique solution $\vec{c} = \vec{0}$, known to imply A^{-1} exists or equivalently $\det(A) \neq 0$. The proof is complete.

Proof of Theorem 29, Orthogonal Vector Test: The proof will be given for $k = 3$, because the details are easily supplied for k vectors, by modifying the displays in the proof. To test independence of the three vectors, we must solve the system of equations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

for the constants c_1, c_2, c_3. This is done for constant c_1 by taking the dot product of the above equation with vector $\vec{v}_1$, to obtain the scalar equation

$$c_1\vec{v}_1 \cdot \vec{v}_1 + c_2\vec{v}_1 \cdot \vec{v}_2 + c_3\vec{v}_1 \cdot \vec{v}_3 = \vec{v}_1 \cdot \vec{0}.$$

Using the orthogonality relations $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$, $\vec{v}_3 \cdot \vec{v}_1 = 0$, then the scalar equation reduces to

$$c_1\vec{v}_1 \cdot \vec{v}_1 + c_2(0) + c_3(0) = 0.$$

Because $\vec{v}_1 \cdot \vec{v}_1 > 0$, then $c_1 = 0$. Symmetrically, vector $\vec{v}_2$ replacing $\vec{v}_1$, the scalar equation becomes

$$c_1(0) + c_2\vec{v}_2 \cdot \vec{v}_2 + c_3(0) = 0.$$

Again, we show $c_2 = 0$. The argument for $c_3 = 0$ is similar. The conclusion: $c_1 = c_2 = c_3 = 0$. The three vectors are independent. The proof is complete.

Proof of Theorem 31, Wronskian Test: The objective of the proof is to solve for the constants $c_1, \ldots, c_n$ in the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \text{for all } x.$$

The functions are proved independent, provided all the constants are zero. The idea of the proof, attributed to Wronski, is to differentiate the above equation $n-1$ times, then substitute $x = x_0$ to obtain a homogeneous $n \times n$ system $A\vec{c} = \vec{0}$ for the components $c_1, \ldots, c_n$ of the vector $\vec{c}$. Because $|A| = W(f_1, \ldots, f_n)(x_0) \neq 0$, the inverse matrix $A^{-1} = \mathbf{adj}(A)/|A|$ exists. Multiply $A\vec{c} = \vec{0}$ on the left by A^{-1} to obtain $\vec{c} = \vec{0}$, completing the proof.

Proof of Theorem 33, Vandermonde Determinant Identity: Let us prove the identity for the case $k = 3$, which serves to simplify notation and displays. Assume distinct samples x_1, x_2, x_3. We hope to establish for $k = 3$ the identity

$$\det(V(x_1, x_2, x_3)) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

The identity is proved from determinant properties, as follows. Replace x_1 by x in the Vandermonde matrix followed by evaluating the determinant. This defines the function $f(x) = \det(A)$, where $A = V(x, x_2, x_3)$. Cofactor expansion along row one of $\det(A)$ reveals that the determinant $f(x) = \det(A)$ is a polynomial in variable x of degree 2:

$$f(x) = (1)\,\mathbf{cof}(A, 1, 1) + (x)\,\mathbf{cof}(A, 1, 2) + (x^2)\,\mathbf{cof}(A, 1, 3).$$

Duplicate rows in a determinant cause it to have zero value, therefore A has determinant zero when we substitute $x = x_2$ or $x = x_3$. Then the quadratic equation $f(x) = 0$ has distinct roots x_2, x_3. The factor theorem of college algebra applies to give two factors $x - x_2$ and $x - x_3$, and then

$$f(x) = c(x_3 - x)(x_2 - x),$$

where c is some constant. Examine the cofactor expansion along the first row in the previous display, then match coefficients of x^2, to show that $c = \mathbf{cof}(A, 1, 3) = (-1)^{1+3}\,\mathbf{minor}(A, 1, 3) = \det(V(x_2, x_3)$. Then

$$f(x) = \det(V(x_2, x_3))(x_3 - x)(x_2 - x).$$

After substitution of $x = x_1$, the equation becomes

$$\det(V(x_1, x_2, x_3)) = \det(V(x_2, x_3))(x_3 - x_1)(x_2 - x_1).$$

The expansion of $\det(V(x_2, x_3)) = \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = x_3 - x_2$ is by Sarrus' Rule. Then

$$\det(V(x_1, x_2, x_3)) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Remarks on Mathematical Induction. An induction argument for the $k \times k$ case proves that

$$\det(V(x_1, x_2, \ldots, x_k)) = \det(V(x_2, \ldots, x_k))\Pi_{j=2}^{k}(x_j - x_1).$$

This is a difficult induction for a novice. The reader should try first to establish the above identity for $k = 4$, by repeating the cofactor expansion step in the 4×4 case. The preceding identity is solved recursively to give the claimed formula for the case $k = 3$:

$$\begin{aligned} \det(V(x_1, x_2, x_3)) &= \det(V(x_2, x_3))[(x_3 - x_1)(x_2 - x_1)] \\ &= \det(V(x_3))(x_3 - x_2)[(x_3 - x_1)(x_2 - x_1)] \\ &= 1 \cdot (x_3 - x_2)(x_3 - x_1)(x_2 - x_1). \end{aligned}$$

The induction proof uses a step like the one below, in which the identity is assumed for all matrix dimensions less than 4:

$$\begin{aligned} \det(V) &= \det(V(x_1, x_2, x_3, x_4)) \\ &= \det(V(x_2, x_3, x_4))[(x_4 - x_1)(x_3 - x_1)(x_2 - x_1)] \\ &= (x_4 - x_3)(x_4 - x_2)(x_3 - x_2)[(x_4 - x_1)(x_3 - x_1)(x_2 - x_1)] \\ &= (x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1). \end{aligned}$$

Exercises 5.4

Scalar and Vector General Solution. Given the scalar general solution of $A\vec{x} = \vec{0}$, find the vector general solution

$$\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots$$

where symbols t_1, t_2, … denote arbitrary constants and $\vec{u}_1, \vec{u}_2, \ldots$ are fixed vectors.

1. $x_1 = 2t_1$, $x_2 = t_1 - t_2$, $x_3 = t_2$

2. $x_1 = t_1 + 3t_2$, $x_2 = t_1$, $x_3 = 4t_2$, $x_4 = t_2$

3. $x_1 = t_1$, $x_2 = t_2$, $x_3 = 2t_1 + 3t_2$

4. $x_1 = 2t_1 + 3t_2 + t_3$, $x_2 = t_1$, $x_3 = t_2$, $x_4 = t_3$

Vector General Solution. Find the vector general solution $\vec{x}$ of $A\vec{x} = \vec{0}$.

5. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 \end{pmatrix}$

10. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

Dimension.

11. Give four examples in $\mathcal{R}^3$ of $S = \mathbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ (3 vectors required) which have respectively dimensions $0, 1, 2, 3$.

12. Give an example in $\mathcal{R}^3$ of 2-dimensional subspaces S_1, S_2 with only the zero vector in common.

13. Let $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$ in abstract vector space V. Explain why $\dim(S) \le 2$.

14. Let $S = \mathbf{span}(\vec{v}_1, \ldots, \vec{v}_k)$ in abstract vector space V. Explain why $\dim(S) \le k$.

15. Let S be a subspace of $\mathcal{R}^3$ with basis $\vec{v}_1, \vec{v}_2$. Define $\vec{v}_3$ to be the **cross product** of $\vec{v}_1, \vec{v}_2$. What is $\dim(\mathbf{span}(\vec{v}_2, \vec{v}_3))$?

16. Let S_1, S_2 be subspaces of $\mathcal{R}^4$ such that $\dim(S_1)\dim(S_2) = 2$. Assume S_1, S_2 have only the zero vector in common. Prove or give a counter-example: the span of the union of S_1, S_2 equals $\mathcal{R}^4$.

Independence in Abstract Spaces.

17. Assume linear combinations of vectors $\vec{v}_1$, $\vec{v}_2$ are uniquely determined, that is, $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$ implies $a_1 = b_1$, $a_2 = b_2$. **Prove** this result: If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, then $c_1 = c_2 = 0$.

18. Assume the zero linear combination of vectors $\vec{v}_1$, $\vec{v}_2$ is uniquely determined, that is, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ implies $c_1 = c_2 = 0$. **Prove** this result: If $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$, then $a_1 = b_1$, $a_2 = b_2$.

19. Prove that two **nonzero** vectors $\vec{v}_1$, $\vec{v}_2$ in an abstract vector space V are independent if and only if $\vec{v}_1$ is not a constant multiple of $\vec{v}_2$.

20. Let $\vec{v}_1$ be a vector in an abstract vector space V. Prove that the one-element set $\vec{v}_1$ is independent if and only if $\vec{v}_1$ is not the zero vector.

21. Let V be an abstract vector space and assume $\vec{v}_1$, $\vec{v}_2$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 2\vec{v}_2$. Prove that $\vec{u}_1$, $\vec{u}_2$ are independent in V.
Advice: Fixed vectors not assumed! Bursting the vector packages is impossible, there are no components.

22. Let V be an abstract vector space and assume $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 4\vec{v}_2$, $\vec{u}_3 = \vec{v}_3 - \vec{v}_1$. Prove that $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ are independent in V.

23. Let S be a finite set of independent vectors in an abstract vector space V. Prove that none of the vectors can be the zero vector.

24. Let S be a finite set of independent vectors in an abstract vector space V. Prove that no vector in the list can be a linear combination of the other vectors.

The Spaces $\mathcal{R}^n$.

25. **(Scalar Multiply)** Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ have components measured in centimeters. Report constants c_1, c_2, c_3 for re-scaled data $c_1\vec{x}$, $c_2\vec{x}$, $c_3\vec{x}$ in units of kilometers, meters and millimeters.

26. **(Matrix Multiply)** Let $\vec{u} = \left(x_1, x_2, x_3, p_1, p_2, p_3 \right)^T$ have position x-units in kilometers and momentum p-units in kilogram-centimeters per millisecond. Determine a matrix M such that the vector $\vec{y} = M\vec{u}$ has SI units of meters and kilogram-meters per second.

27. Let $\vec{v}_1$, $\vec{v}_2$ be two independent vectors in $\mathcal{R}^n$. Assume $c_1\vec{v}_1 + c_2\vec{v}_2$ lies strictly interior to the parallelogram determined by $\vec{v}_1$, $\vec{v}_2$. Give geometric details explaining why $0 < c_1 < 1$ and $0 < c_2 < 1$.

28. Prove the 4 scalar multiply toolkit properties for fixed vectors in $\mathcal{R}^3$.

29. Define

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, -\vec{v} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}.$$

Prove the 4 addition toolkit properties for fixed vectors in $\mathcal{R}^3$.

30. Use the 8 property toolkit in $\mathcal{R}^3$ to prove that zero times a vector is the zero vector.

31. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be a basis for $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a basis for $\mathcal{R}^3$.

32. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be dependent in $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a dependent set in $\mathcal{R}^3$.

Digital Photographs. Let V be the vector space of all 2×3 matrices. A matrix in V is a 6-pixel digital photo, a sub-section of a larger photo.

Replace one zero in the 2×3 zero matrix with a one. The 6 answers B_1, $\ldots$, B_6 are numbered by $B_j[n, m] = 1$ when $3(n-1) + m = j$.

33. Prove that $B_1, \ldots, B_6$ are independent and span V: they are a **basis** for V. Each B_i **lights up** one pixel in the 2×3 sub-photo.

34. Define red, green and blue monochrome matrices R, G, B by

$$\begin{pmatrix} 2\,0\,0 \\ 0\,1\,1 \\ 5\,8\,1 \end{pmatrix}, \begin{pmatrix} 3\,0\,0 \\ 0\,4\,0 \\ 0\,1\,0 \end{pmatrix}, \begin{pmatrix} 5\,0\,0 \\ 0\,3\,0 \\ 1\,0\,5 \end{pmatrix}.$$

Define base $x = 16$. Compute $A = R + xG + x^2B$.

35. Let $A = 2\begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \end{pmatrix} + 4\begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \end{pmatrix}$. Assume a black and white image. Describe photo A, assuming 0 means black.

36. Let c_1, c_2 be integer encodings of RGB intensities. Describe the photo $A = c_1B_1 + c_2B_3$.

Polynomial Spaces. Let V be the vector space of all cubic or less polynomials $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$.

37. Find a subspace S of V, $\dim(S) = 2$, which contains the vector $1 + x$.

38. Let S be the subset of V spanned by x, x^2 and x^3. Prove that S is a subspace of V which does not contain the polynomial $1 + x$.

39. Define set S by the conditions $p(0) = 0, p(1) = 0$. Find a basis for S.

40. Define set S by the condition $p(0) = \int_0^1 p(x)dx$. Find a basis for S.

The Space $C(E)$. Define $\vec{f}$ to be the vector package with domain $E = \{x \ : \ -1 \le x \le 1\}$ and equation $y = |x|$. Similarly, $\vec{g}$ is defined by equation $y = x$.

41. Show independence of $\vec{f}, \vec{g}$.

42. Find the dimension of $\mathbf{span}(\vec{f}, \vec{g})$.

43. Let $h(x) = 0$ on $-1 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 1$. Show that $\vec{h}$ is in $C(E)$.

44. Let $h(x) = -1$ on $-1 \le x \le 0$, $h(x) = 1$ on $0 \le x \le 1$. Show that $\vec{h}$ is not in $C(E)$.

45. Let $h(x) = 0$ on $-1 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 1$. Show that $\vec{h}$ is in $\mathbf{span}(\vec{f}, \vec{g})$.

46. Let $h(x) = \tan(\pi x/2)$ on $-1 < x < 1$, $h(1) = h(-1) = 0$. Explain why $\vec{h}$ is not in $C(E)$

The Space $C^1(E)$. Define $\vec{f}$ to be the vector package with domain $E = \{x : -1 \le x \le 1\}$ and equation $y = x|x|$. Similarly, $\vec{g}$ is defined by equation $y = x^2$.

47. Verify that $\vec{f}$ is in $C^1(E)$, but its derivative is not.

48. Show that $\vec{f}, \vec{g}$ are independent in $C^1(E)$.

The Space $C^k(E)$. Let E be the unit interval $0 \le x \le 1$ and define $\vec{f}$ to be domain E plus the equation $y = e^{-x^2}$

49. Justify that $\vec{f}$ belongs to $C^k(E)$ for all $k \ge 1$.

50. Compute the first three derivatives of $\vec{f}$ at $x = 0$.

Solution Space. A differential equations solver finds general solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$. Use vector space $V = C^4(E)$ where E is the whole real line.

51. Write the solution set S as the span of four vectors in V.

52. Find a basis for the solution space S of the differential equation. Verify independence using the sampling test or Wronskian test.

53. Find a differential equation $y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x$.

54. Find a differential equation $y'''' + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Algebraic Independence Test for Two Vectors. Solve for c_1, c_2 in the independence test for two vectors, showing all details.

55. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

56. $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Dependence of two vectors. Solve for c_1, c_2 not both zero in the independence test for two vectors, showing all details for dependency of the two vectors.

57. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

58. $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

Independence Test for Three Vectors. Solve for the constants c_1, c_2, c_3 in the relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Report dependent of independent vectors. If dependent, then display a dependency relation.

59. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

60. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Independence in an Abstract Vector Space. In vector space V, report independence or a dependency relation for the given vectors.

61. Space $V = C(-\infty, \infty)$, $\vec{v}_1 = 1 + x$, $\vec{v}_2 = 2 + x$, $\vec{v}_3 = 3 + x^2$.

62. Space $V = C(-\infty, \infty)$, $\vec{v}_1 = x^{3/5}$, $\vec{v}_2 = x^2$, $\vec{v}_3 = 2x^2 + 3x^{3/5}$

63. Space V is all 3×3 matrices. Let $\vec{v}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 2 & 5 \\ 0 & 3 & 5 \end{pmatrix}$.

64. Space V is all 2×2 matrices. Let $\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$.

Rank Test. Compute the rank of the augmented matrix to determine independence or dependence of the given vectors.

65. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

66. $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Determinant Test. Evaluate the determinant of the augmented matrix to determine independence or dependence of the given vectors.

67. $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$

68. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Sampling Test for Functions. Invent samples to verify independence.

69. $\cosh(x), \sinh(x)$

70. $x^{7/3}, x\sin(x)$

71. $1, x, \sin(x)$

72. $1, \cos^2(x), \sin(x)$

Sampling Test and Dependence. For three functions f_1, f_2, f_3 to be dependent, constants c_1, c_2, c_3 must be found such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

The trick is that c_1, c_2, c_3 are not all zero and the relation holds **for all** x. The sampling test method can discover the constants, but it is **unable to prove dependence**!

73. Functions $1, x, 1+x$ are dependent. Insert $x = 0, 1, 2$ and solve for c_1, c_2, c_3, to discover a dependency relation.

74. Functions $1, \cos^2(x), \sin^2(x)$ are dependent. Cleverly choose 3 values of x, insert them, then solve for c_1, c_2, c_3, to discover a dependency relation.

Vandermonde Determinant.

75. Let $V = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = x_2 - x_1.$$

76. Let $V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Wronskian Test for Functions. Apply the Wronskian Test to verify independence.

77. $\cos(x), \sin(x)$.

78. $\cos(x), \sin(x), \sin(2x)$.

79. $x, x^{5/3}$.

80. $\cosh(x), \sinh(x)$.

Wronskian Test: Theory.

81. The functions x^2 and $x|x|$ are continuously differentiable and have zero Wronskian. Verify that they **fail to be dependent** on $-1 < x < 1$.

82. The Wronskian Test can verify the independence of the powers $1, x, \ldots, x^k$. Show the determinant details.

Extracting a Basis. Given a list of vectors in space $V = \mathcal{R}^4$, extract a largest independent subset.

83. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

84. $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Extracting a Basis. Given a list of vectors in space $V = C(-\infty, \infty)$, extract a largest independent subset.

85. $x, x\cos^2(x), x\sin^2(x), e^x, x + e^x$

86. $1, 2 + x, \frac{x}{1+x^2}, \frac{x^2}{1+x^2}$

Euler Solution Atom. Identify the Euler solution atoms in the given list. Strictly apply the definition: e^x is an atom but $2e^x$ is not.

87. $1, 2 + x, e^{2.15x}, e^{x^2}, \frac{x}{1+x^2}$

88. $2, x^3, e^{x/\pi}, e^{2x+1}, \ln|1 + x|$

Euler Solution Atom Test. Establish independence of set S_1.
Suggestion: First establish an identity $\mathbf{span}(S_1) = \mathbf{span}(S_2)$, where S_2 is an invented list of distinct atoms. The Test implies S_2 is independent. Extract a largest independent subset of S_1, using independence of S_2.

89. Set S_1 is the list $2, 1 + x^2, 4 + 5e^x, \pi e^{2x+\pi}, 10x\cos(x)$.

90. Set S_1 is the list $1 + x^2, 1 - x^2, 2\cos(3x), \cos(3x) + \sin(3x)$.