**Week 1 Examples**

**Three Examples.** Solve differential equations without a book. Three basic examples used throughout a course in differential equations, which require only a calculus background.

**Growth-Decay:** \( \frac{dA(t)}{dt} = kA(t), \quad A(0) = A_0. \)

The unique solution is \( A(t) = A_0 e^{kt}. \) Radioactive decay. Jeweler’s bench light experiment. Malthusian population dynamics. RC and LR circuits. Drug elimination. First-order chemical reactions, law of mass-action. Compound continuous bank interest.

**Newton Cooling:** \( \frac{du(t)}{dt} = -h(u(t) - u_1), \quad u(0) = u_0. \)

The solution is \( u(t) = u_1 + (u_0 - u_1) e^{-ht}. \) Hot chocolate at initial temperature \( u_0 \) with room thermometer reading \( u_1. \) Symbol \( u(t) = \) time-varying hot chocolate dial thermometer temperature.

**Verhulst Dynamics:** \( \frac{dP(t)}{dt} = (a - bP(t))P(t), \quad P(0) = P_0. \)

The solution is \( P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \) Fish population \( P(t) \) in Cecret Lake at Alta. Carrying capacity. Stocking and re-stocking. Harvesting.

**Example 1:** Exercise 1.2-2: Solve \( dy/dx = (x - 2)^2, \quad y(2) = 1. \)

Method of quadrature. Answer Check details. Non-reversible steps, false proof for \( 0 = 1. \)


**Example 2:** Exercise 1.2-4: \( dy/dx = 1/x^2, \quad y(1) = 5. \)

Power rule in Newton calculus. Answer check shortcuts.

**Example 3:** Exercise 1.2-10: \( dy/dx = x e^{-x}, \quad y(0) = 1. \)

Integral tables and integration by parts. Jennifer Lahti’s solution:


**Example 4:** Exercise 1.3-8: \( dy/dx = x^2 - y \)

Thread edge-to-edge solutions through the direction field at each blue dot. JPEG image source:


**Example 5:** Exercise 1.3-14: \( dy/dx = y^{1/3}, \quad y(0) = 0 \)

Explain application of the Peano and Picard theorems.

Computer numerical methods fail on this example. Why?

**Example 6:** Exercise 1.4-6: Solve \( y' = 3\sqrt{xy} \)

Three answers. Book reports only one answer.

**Example 7:** Exercise 1.4-10: Solve \( (1 + x^2)y' = (1 + y)^2 \)

Two answers. Book reports only one answer.

**Example 8:** Exercise 1.4-18: Solve \( x^2y' = 1 - x^2 + y^2 - x^2y^2 \) [See Example 11 infra]

**Example 9:** Exercise 1.4-22: Solve \( y' = 4x^3y - y, \quad y(1) = -3 \)

**Example 10:** Show that \( y' = x + y \) is not separable. TEST I. \( f_x/f \) depends on \( y \) implies \( y' = f(x, y) \) not separable.

**Example 11:** Find a factorization \( f(x, y) = F(x)G(y) \) given

(1) \( f(x, y) = 2xy + 4y + 3x + 6 \)

(2) \( f(x, y) = (1 - x^2 + y^2 - x^2y^2)/x^2 \)

Answers: (1) \( F = x + 2, \quad G = 2y + 3; \) (2) \( F = (1 - x^2)/x^2, \quad G = 1 + y^2. \) Main idea: Choose \( y = 0 \) in \( F(x) = f(x, y)/G(y) \) to find \( F(x) = (3x + 6)/G(0) \) in equation (1). How to find \( G? \) Warning: Divide by zero is not allowed. Choose \( y = 0, \quad y = 1, \) etc, until no divide by zero error.

**Example 12:** Midterm 1 examples: \( y' = x + y, \quad y' = x + y^2, \quad y' = x^2 + y^2 \)