# Eigenanalysis

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## What's Eigenanalysis?

Matrix eigenanalysis is a computational theory for the matrix equation

$$\vec{\mathbf{y}} = A\vec{\mathbf{x}}.$$

## Fourier's Eigenanalysis Model

For exposition purposes, we assume $A$ is a $3 \times 3$ matrix.

$$
\begin{aligned}
\vec{\mathbf{x}} &= c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 \text{ implies} \\
\vec{\mathbf{y}} &= A\vec{\mathbf{x}} \\
&= c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3.
\end{aligned} \tag{1}
$$

## Eigenanalysis Notation

The scale factors $\lambda_1$, $\lambda_2$, $\lambda_3$ and independent vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ depend only on $A$. Symbols $c_1$, $c_2$, $c_3$ stand for arbitrary numbers. This implies variable $\vec{\mathbf{x}}$ exhausts all possible fixed vectors in $R^3$.

**Fourier's Model is a Replacement Process**

$$A\left(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3\right) = c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2 + c_3\lambda_3\vec{\mathbf{v}}_3.$$

To compute $A\vec{\mathbf{x}}$ from $\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$, replace each vector $\vec{\mathbf{v}}_i$ by its scaled version $\lambda_i\vec{\mathbf{v}}_i$.

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Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (1). Fourier's model is known to fail for certain matrices $A$.

## Powers and Fourier's Model

Equation (1) applies to compute powers $\boldsymbol{A}^n$ of a matrix $\boldsymbol{A}$ using only the basic vector space toolkit. To illustrate, only the vector toolkit for $\boldsymbol{R}^3$ is used in computing

$$\boldsymbol{A}^5\vec{\mathbf{x}} = x_1\lambda_1^5\vec{\mathbf{v}}_1 + x_2\lambda_2^5\vec{\mathbf{v}}_2 + x_3\lambda_3^5\vec{\mathbf{v}}_3.$$

This calculation does not depend upon finding previous powers $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, $\boldsymbol{A}^4$ as would be the case by using matrix multiply.

## Details for $A^3(\vec{\mathbf{x}})$

Let $\vec{\mathbf{x}} = x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3$. Then

$$\begin{aligned} A^3(\vec{\mathbf{x}}) &= A^2(A(\vec{\mathbf{x}})) \\ &= A^2(x_1\lambda_1\vec{\mathbf{v}}_1 + x_2\lambda_2\vec{\mathbf{v}}_2 + x_3\lambda_3\vec{\mathbf{v}}_3) \\ &= A(A(x_1\lambda_1\vec{\mathbf{v}}_1 + x_2\lambda_2\vec{\mathbf{v}}_2 + x_3\lambda_3\vec{\mathbf{v}}_3)) \\ &= A(x_1\lambda_1^2\vec{\mathbf{v}}_1 + x_2\lambda_2^2\vec{\mathbf{v}}_2 + x_3\lambda_3^2\vec{\mathbf{v}}_3) \\ &= x_1\lambda_1^3\vec{\mathbf{v}}_1 + x_2\lambda_2^3\vec{\mathbf{v}}_2 + x_3\lambda_3^3\vec{\mathbf{v}}_3 \end{aligned}$$

## Differential Equations and Fourier's Model

Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{array}{rcrcrcr} x_1' &=& x_1 &+& 3x_2, && \\ x_2' &=& && 2x_2 &-& x_3, \\ x_3' &=& && &-& 5x_3. \end{array}$$

The general solution is given by the formula [Fourier's theorem, proved later]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formulas

$$\begin{array}{rcl} \vec{\mathbf{x}}(0) &=& c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 \\ && \text{undergoes replacements } \vec{\mathbf{v}}_i \longrightarrow e^{\lambda_i t}\vec{\mathbf{v}}_i \text{ to obtain} \\ \vec{\mathbf{x}}(t) &=& c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3. \end{array}$$

## Fourier's model illustrated

Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\boldsymbol{\lambda}_1 = 1, \qquad \boldsymbol{\lambda}_2 = 2, \qquad \boldsymbol{\lambda}_3 = -5,$$

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\begin{aligned} \vec{\mathbf{x}} &= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies} \\ \boldsymbol{A}\vec{\mathbf{x}} &= c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \end{aligned}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\vec{\mathbf{y}} = \boldsymbol{A}\vec{\mathbf{x}}$ by scaling independent vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, which is a triad or **coordinate system**.

**What is Eigenanalysis?**

The subject of **eigenanalysis** discovers a coordinate system $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and scale factors $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ such that Fourier's model holds. Fourier's model simplifies the matrix equation $\vec{\mathbf{y}} = \boldsymbol{A}\vec{\mathbf{x}}$, through the formula

$$\boldsymbol{A}(\boldsymbol{c}_1\vec{\mathbf{v}}_1 + \boldsymbol{c}_2\vec{\mathbf{v}}_2 + \boldsymbol{c}_3\vec{\mathbf{v}}_3) = \boldsymbol{c}_1\boldsymbol{\lambda}_1\vec{\mathbf{v}}_1 + \boldsymbol{c}_2\boldsymbol{\lambda}_2\vec{\mathbf{v}}_2 + \boldsymbol{c}_3\boldsymbol{\lambda}_3\vec{\mathbf{v}}_3.$$

## What's an Eigenvalue?

It is a **scale factor**. An eigenvalue is also called a *proper value* or a *hidden value* or a *characteristic value*. Symbols $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3$ used in Fourier's model are eigenvalues.

> The **eigenvalues** of a model are scale factors. Think of them as a system of units *hidden* in the matrix $\boldsymbol{A}$.

## What's an Eigenvector?

Symbols $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors* or *characteristic vectors*. They are assumed independent.

> The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues). Think of each eigenpair $(\boldsymbol{\lambda}, \vec{\mathbf{v}})$ as a coordinate axis $\vec{\mathbf{v}}$ where the action of matrix $\boldsymbol{A}$ is to move $\boldsymbol{\lambda}$ units along $\vec{\mathbf{v}}$.

## Data Conversion Example

Let $\vec{\mathbf{x}}$ in $\boldsymbol{R}^3$ be a data set variable with coordinates $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{x_3}$ recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit $\vec{\mathbf{x}}$-data into proper MKS units (meters-kilogram-second) $\vec{\mathbf{y}}$-data via the equations

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= 0.001 x_2, \\ y_3 &= 0.01 x_3. \end{aligned} \tag{2}$$

Equations (2) are a **model** for changing units. Scaling factors $\boldsymbol{\lambda_1 = 1}$, $\boldsymbol{\lambda_2 = 0.001}$, $\boldsymbol{\lambda_3 = 0.01}$ are the **eigenvalues** of the model.

## Data Conversion Example – Continued

Problem (2) can be represented as $\vec{\mathbf{y}} = \boldsymbol{A}\vec{\mathbf{x}}$, where the diagonal matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{\lambda}_1 & 0 & 0 \\ 0 & \boldsymbol{\lambda}_2 & 0 \\ 0 & 0 & \boldsymbol{\lambda}_3 \end{pmatrix}, \quad \boldsymbol{\lambda}_1 = 1, \ \boldsymbol{\lambda}_2 = \frac{1}{1000}, \ \boldsymbol{\lambda}_3 = \frac{1}{100}.$$

Fourier's model for this matrix $\boldsymbol{A}$ is

$$\boldsymbol{A}\left(\boldsymbol{c}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{c}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{c}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \boldsymbol{c}_1\boldsymbol{\lambda}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{c}_2\boldsymbol{\lambda}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{c}_3\boldsymbol{\lambda}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The **eigenvectors** $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ of the model are the columns of the identity matrix.

**Summary**

The **eigenvalues** of a model are **scale factors**, normally represented by symbols

$$\boldsymbol{\lambda}_1, \quad \boldsymbol{\lambda}_2, \quad \boldsymbol{\lambda}_3, \quad \ldots.$$

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues). They are normally represented by symbols

$$\vec{\mathbf{v}}_1, \quad \vec{\mathbf{v}}_2, \quad \vec{\mathbf{v}}_3, \quad \ldots.$$

**History of Fourier's Model**

The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\vec{y} = A\vec{x}$.

> The vector $\vec{y} = A\vec{x}$ is obtained from eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \ldots, \lambda_n\vec{v}_n$:
>
> $$\begin{aligned} \vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \quad \text{implies} \\ \vec{y} &= x_1\lambda_1\vec{v}_1 + x_2\lambda_2\vec{v}_2 + \cdots + c_n\lambda_n\vec{v}_n. \end{aligned}$$

## Determining Equations

The eigenvalues and eigenvectors are determined by homogeneous matrix–vector equations. In Fourier's model

$$A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3$$

choose $c_1 = 1$, $c_2 = c_3 = 0$. The equation reduces to $A\vec{v}_1 = \lambda_1\vec{v}_1$. Similarly, taking $c_1 = c_2 = 0$, $c_2 = 1$ implies $A\vec{v}_2 = \lambda_2\vec{v}_2$. Finally, taking $c_1 = c_2 = 0$, $c_3 = 1$ implies $A\vec{v}_3 = \lambda_3\vec{v}_3$. This proves the following fundamental result.

**Theorem 1 (Determining Equations in Fourier's Model)**
Assume Fourier's model holds. Then the eigenvalues and eigenvectors are determined by the three equations

$$\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1, \\ A\vec{v}_2 &= \lambda_2\vec{v}_2, \\ A\vec{v}_3 &= \lambda_3\vec{v}_3. \end{aligned}$$

## Determining Equations and Linear Algebra

The three relations of the theorem can be distilled into one homogeneous matrix–vector equation

$$\boldsymbol{A}\vec{\mathbf{v}} = \boldsymbol{\lambda}\vec{\mathbf{v}}.$$

Write it as $\boldsymbol{A}\vec{\mathbf{x}} - \boldsymbol{\lambda}\vec{\mathbf{x}} = \vec{\mathbf{0}}$, then replace $\boldsymbol{\lambda}\vec{\mathbf{x}}$ by $\boldsymbol{\lambda I}\vec{\mathbf{x}}$ to obtain the standard form[a]

$$(\boldsymbol{A} - \boldsymbol{\lambda I})\vec{\mathbf{v}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{v}} \neq \vec{\mathbf{0}}.$$

Let $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda I}$. The equation $\boldsymbol{B}\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has a nonzero solution $\vec{\mathbf{v}}$ if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occurs if and only if $\mathbf{rref}(\boldsymbol{B})$ has a row of zeros. Determinant theory gives a more concise statement: $\mathbf{det}(\boldsymbol{B}) = \mathbf{0}$ if and only if $\boldsymbol{B}\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has infinitely many solutions. This proves the following result.

[a] Identity $\boldsymbol{I}$ is required to factor out the matrix $\boldsymbol{A} - \boldsymbol{\lambda I}$. It is wrong to factor out $\boldsymbol{A} - \boldsymbol{\lambda}$, because $\boldsymbol{A}$ is $\mathbf{3} \times \mathbf{3}$ and $\boldsymbol{\lambda}$ is $\mathbf{1} \times \mathbf{1}$, incompatible sizes for matrix addition.

**Theorem 2 (Characteristic Equation)**
If Fourier's model holds, then the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ are roots $\lambda$ of the polynomial equation

$$\det(A - \lambda I) = 0.$$

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation**. The **characteristic polynomial** is the polynomial on the left, $\det(A - \lambda I)$, normally obtained by cofactor expansion or the triangular rule.

## Eigenvectors and Toolkit Sequences

Eigenpairs of $\boldsymbol{A}$ are found using the linear algebra toolkit of swap, combo, multiply.

**Theorem 3 (Finding Eigenvectors of $A$)**

For each root $\boldsymbol{\lambda}$ of the characteristic equation, write the toolkit sequence for $\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\lambda I}$, ending with $\mathbf{rref}(\boldsymbol{B})$ Solve for the general solution $\vec{\mathbf{v}}$ of the homogeneous equation $\boldsymbol{B}\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Solution $\vec{\mathbf{v}}$ uses invented symbols $\boldsymbol{t_1}$, $\boldsymbol{t_2}$, .... The vector basis answers $\boldsymbol{\partial_{t_1}}\vec{\mathbf{v}}$, $\boldsymbol{\partial_{t_2}}\vec{\mathbf{v}}$, ... are independent **eigenvectors** of $\boldsymbol{A}$ paired to eigenvalue $\boldsymbol{\lambda}$. These vectors are known in linear algebra as *Strang's Special Solutions*.

**Proof**: The equation $\boldsymbol{A}\vec{\mathbf{v}} = \boldsymbol{\lambda}\vec{\mathbf{v}}$ is equivalent to $\boldsymbol{B}\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Because $\det(\boldsymbol{B}) = \mathbf{0}$, then this system has infinitely many solutions, which implies the toolkit sequence starting at $\boldsymbol{B}$ ends with $\mathbf{rref}(\boldsymbol{B})$ having at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots$, and then the vector basis is obtained by from the corresponding list of partial derivatives. Each basis element is a nonzero solution of $\boldsymbol{A}\vec{\mathbf{v}} = \boldsymbol{\lambda}\vec{\mathbf{v}}$. By construction, the basis elements (eigenvectors for $\boldsymbol{\lambda}$) are collectively independent. The proof is complete.

## Eigenpairs of a Matrix

**Definition 1 (Eigenpair)**
An **eigenpair** is an eigenvalue $\boldsymbol{\lambda}$ together with a matching eigenvector $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ satisfying the equation $\boldsymbol{A}\vec{\mathbf{v}} = \boldsymbol{\lambda}\vec{\mathbf{v}}$. The pairing implies that scale factor $\boldsymbol{\lambda}$ is applied to direction $\vec{\mathbf{v}}$.

An applied view of an eigenpair is a coordinate axis $\vec{\mathbf{v}}$ and a unit system along this axis. The action of the matrix $\boldsymbol{A}$ is to move $\boldsymbol{\lambda}$ units along this axis.

A $\mathbf{3} \times \mathbf{3}$ matrix $\boldsymbol{A}$ for which Fourier's model holds has eigenvalues $\boldsymbol{\lambda_1}$, $\boldsymbol{\lambda_2}$, $\boldsymbol{\lambda_3}$ and corresponding eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$. The **eigenpairs** of $\boldsymbol{A}$ are

$$(\boldsymbol{\lambda}_1, \vec{\mathbf{v}}_1)\,, (\boldsymbol{\lambda}_2, \vec{\mathbf{v}}_2)\,, (\boldsymbol{\lambda}_3, \vec{\mathbf{v}}_3)\,.$$

## Eigenvectors are Independent

**Theorem 4 (Independence of Eigenvectors)**
If $(\boldsymbol{\lambda}_1, \vec{\mathbf{v}}_1)$ and $(\boldsymbol{\lambda}_2, \vec{\mathbf{v}}_2)$ are two eigenpairs of $\boldsymbol{A}$ and $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$, then $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent.
More generally, if $(\boldsymbol{\lambda}_1, \vec{\mathbf{v}}_1)$, ..., $(\boldsymbol{\lambda}_k, \vec{\mathbf{v}}_k)$ are eigenpairs of $\boldsymbol{A}$ corresponding to distinct eigenvalues $\boldsymbol{\lambda}_1$, ..., $\boldsymbol{\lambda}_k$, then $\vec{\mathbf{v}}_1$, ..., $\vec{\mathbf{v}}_k$ are independent.

**Theorem 5 (Distinct Eigenvalues)**
If an $\boldsymbol{n} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$ has $\boldsymbol{n}$ distinct eigenvalues, then its eigenpairs $(\boldsymbol{\lambda}_1, \vec{\mathbf{v}}_1)$, ..., $(\boldsymbol{\lambda}_n, \vec{\mathbf{v}}_n)$ produce independent eigenvectors $\vec{\mathbf{v}}_1$, ..., $\vec{\mathbf{v}}_n$. Therefore, Fourier's model holds:

$$\boldsymbol{A}\left(\sum_{i=1}^{n} \boldsymbol{c}_i \vec{\mathbf{v}}_i\right) = \sum_{i=1}^{n} \boldsymbol{c}_i (\boldsymbol{\lambda}_i \vec{\mathbf{v}}_i).$$