What's Eigenanalysis?

Matrix eigenanalysis is a computational theory for the matrix equation $\mathbf{y} = \boldsymbol{A}\mathbf{x}$. For exposition purposes, we assume $\boldsymbol{A}$ is a $\mathbf{3} \times \mathbf{3}$ matrix.

Fourier's Eigenanalysis Model

$$\begin{aligned} \mathbf{x} &= \boldsymbol{c}_1\mathbf{v}_1 + \boldsymbol{c}_2\mathbf{v}_2 + \boldsymbol{c}_3\mathbf{v}_3 \text{ implies} \\ \mathbf{y} &= \boldsymbol{A}\mathbf{x} \\ &= \boldsymbol{c}_1\boldsymbol{\lambda}_1\mathbf{v}_1 + \boldsymbol{c}_2\boldsymbol{\lambda}_2\mathbf{v}_2 + \boldsymbol{c}_3\boldsymbol{\lambda}_3\mathbf{v}_3. \end{aligned} \tag{1}$$

The scale factors $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ and independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ depend only on $\boldsymbol{A}$. Symbols $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, $\boldsymbol{c}_3$ stand for arbitrary numbers. This implies variable $\mathbf{x}$ exhausts all possible $\mathbf{3}$-vectors in $\boldsymbol{R}^3$.

Fourier's model is a replacement process

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

To compute $A\mathbf{x}$ from $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, replace each vector $\mathbf{v}_i$ by its scaled version $\lambda_i\mathbf{v}_i$.

Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (1). Fourier's model is known to fail for certain matrices A.

Powers and Fourier's Model

Equation (1) applies to compute powers $\boldsymbol{A}^n$ of a matrix $\boldsymbol{A}$ using only the basic vector space toolkit. To illustrate, only the vector toolkit for $\boldsymbol{R}^3$ is used in computing

$$\boldsymbol{A}^5\mathbf{x} = x_1\lambda_1^5\mathbf{v}_1 + x_2\lambda_2^5\mathbf{v}_2 + x_3\lambda_3^5\mathbf{v}_3.$$

This calculation does not depend upon finding previous powers $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, $\boldsymbol{A}^4$ as would be the case by using matrix multiply.

Differential Equations and Fourier's Model

Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{aligned} x_1' &= x_1 + 3x_2, \\ x_2' &= \phantom{x_1 + {}} 2x_2 - x_3, \\ x_3' &= - 5x_3. \end{aligned}$$

The general solution is given by the formula [Fourier's theorem, proved later]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3.$$

Fourier's model illustrated

Let

$$
\begin{aligned}
\boldsymbol{A} &= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix} \\
\boldsymbol{\lambda_1} &= \mathbf{1}, \qquad \boldsymbol{\lambda_2} = \mathbf{2}, \qquad \boldsymbol{\lambda_3} = \mathbf{-5}, \\
\mathbf{v_1} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.
\end{aligned}
$$

Then Fourier's model holds (details later) and

$$
\begin{aligned}
\mathbf{x} &= \boldsymbol{c_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{c_2} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{c_3} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies} \\
\boldsymbol{A}\mathbf{x} &= \boldsymbol{c_1}(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{c_2}(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{c_3}(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}
\end{aligned}
$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ by scaling independent vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, which is a triad or **coordinate system**.

What is Eigenanalysis?

The subject of **eigenanalysis** discovers a coordinate system and scale factors such that Fourier's model holds. Fourier's model simplifies the matrix equation $\mathbf{y} = \boldsymbol{A}\mathbf{x}$, through the formula

$$\boldsymbol{A}(\boldsymbol{c}_1\mathbf{v}_1 + \boldsymbol{c}_2\mathbf{v}_2 + \boldsymbol{c}_3\mathbf{v}_3) = \boldsymbol{c}_1\boldsymbol{\lambda}_1\mathbf{v}_1 + \boldsymbol{c}_2\boldsymbol{\lambda}_2\mathbf{v}_2 + \boldsymbol{c}_3\boldsymbol{\lambda}_3\mathbf{v}_3.$$

What's an Eigenvalue?

It is a scale factor. An eigenvalue is also called a *proper value* or a *hidden value*. Symbols $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_3$ used in Fourier's model are eigenvalues.

What's an Eigenvector?

Symbols $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors*. They are assumed independent.

> The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

A Key Example

Let x in R^3 be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit x-data into proper MKS units (meters-kilogram-second) y-data via the equations

$$
\begin{aligned}
y_1 &= x_1, \\
y_2 &= 0.001x_2, \\
y_3 &= 0.01x_3.
\end{aligned}
\tag{2}
$$

Equations (2) are a **model** for changing units. Scaling factors $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 0.01$ are the **eigenvalues** of the model. To summarize:

> The **eigenvalues** of a model are **scale factors**, normally represented by symbols $\lambda_1, \lambda_2, \lambda_3, \ldots$.

Data Conversion Example – Continued

Problem (2) can be represented as $\mathbf{y} = \boldsymbol{A}\mathbf{x}$, where the diagonal matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{\lambda}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}_3 \end{pmatrix}, \quad \boldsymbol{\lambda}_1 = \mathbf{1}, \; \boldsymbol{\lambda}_2 = \frac{\mathbf{1}}{\mathbf{1000}}, \; \boldsymbol{\lambda}_3 = \frac{\mathbf{1}}{\mathbf{100}}.$$

Fourier's model for this matrix $\boldsymbol{A}$ is

$$\boldsymbol{A}\left(\boldsymbol{c}_1 \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{c}_2 \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{c}_3 \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}\right) = \boldsymbol{c}_1\boldsymbol{\lambda}_1 \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{c}_2\boldsymbol{\lambda}_2 \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{c}_3\boldsymbol{\lambda}_3 \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

1 Example (Computing 3×3 Eigenpairs)

Find all eigenpairs of the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

College Algebra

The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= ((1-\lambda)^2 + 4)(3-\lambda) && \text{Cofactor rule and Sarrus' rule.} \end{aligned}$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1 - \lambda)^2 + 4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

Linear Algebra

The eigenpairs are

$$\left(\mathbf{1+2\boldsymbol{i}}, \begin{pmatrix} -\boldsymbol{i} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}\right), \left(\mathbf{1-2\boldsymbol{i}}, \begin{pmatrix} \boldsymbol{i} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}\right), \left(\mathbf{3}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}\right).$$

Details appear below.

Eigenvector v_1 for $\lambda_1 = 1 + 2i$

$$B = A - \lambda_1 I$$

$$= \begin{pmatrix} 1-\lambda_1 & 2 & 0 \\ -2 & 1-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{pmatrix}$$

$$\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Multiply rule.

$$\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Combination, factor=$-i$.

$$\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Swap rule.

$$= \mathrm{rref}(A - \lambda_1 I)$$ Reduced echelon form.

The partial derivative $\partial_{t_1}\mathrm{v}$ of the general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector

$$\mathrm{v}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

Eigenvector $\mathbf{v}_2$ for $\boldsymbol{\lambda}_2 = 1 - 2i$

The problem $(\boldsymbol{A} - \boldsymbol{\lambda}_2 \boldsymbol{I})\mathbf{v}_2 = \mathbf{0}$ has solution $\mathbf{v}_2 = \overline{\mathbf{v}_1}$.

To see why, take conjugates across the equation to give $(\overline{\boldsymbol{A}} - \overline{\boldsymbol{\lambda}_2} \boldsymbol{I})\overline{\mathbf{v}_2} = \mathbf{0}$. Then $\overline{\boldsymbol{A}} = \boldsymbol{A}$ ($\boldsymbol{A}$ is real) and $\boldsymbol{\lambda}_1 = \overline{\boldsymbol{\lambda}_2}$ gives $(\boldsymbol{A} - \boldsymbol{\lambda}_1 \boldsymbol{I})\overline{\mathbf{v}_2} = \mathbf{0}$. Then $\overline{\mathbf{v}_2} = \mathbf{v}_1$.

Finally,

$$\mathbf{v}_2 = \overline{\overline{\mathbf{v}}}_2 = \overline{\mathbf{v}_1} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

Eigenvector v_3 for $\lambda_3 = 3$

$$
\begin{aligned}
A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 \\ -2 & 1-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\
&= \mathrm{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
\end{aligned}
$$

The partial derivative $\partial_{t_1}\mathbf{v}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$