Linear Dynamical Systems
Matrix Exponential: Putzer Formula for $e^{At}$
Variation of Parameters and Undetermined Coefficients

- The $2 \times 2$ Matrix Exponential $e^{At}$
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The $2 \times 2$ Matrix Exponential $e^{At}$

**Definition.** The matrix exponential $e^{At}$ is the $n \times n$ matrix $\Phi(t)$ defined by

$$
\begin{align*}
(1) \quad & \frac{d}{dt} \Phi = A\Phi, \\
(2) \quad & \Phi(0) = I.
\end{align*}
$$

Alternatively, $\Phi$ is the augmented matrix of solution vectors for the $n$ problems $\frac{d}{dt} \vec{v}_k = A\vec{v}_k$, $\vec{v}_k(0) =$ column $k$ of $I$, $1 \leq k \leq n$.

**Example.** A $2 \times 2$ matrix $A$ has exponential matrix $e^{At}$ with columns equal to the solutions of the two problems

$$
\begin{align*}
\begin{cases}
\frac{d}{dt} \vec{v}_1(t) = A\vec{v}_1(t), \\
\vec{v}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{cases} & \quad \begin{cases}
\frac{d}{dt} \vec{v}_2(t) = A\vec{v}_2(t), \\
\vec{v}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{cases}
\end{align*}
$$

Briefly, the $2 \times 2$ matrix $\Phi(t) = e^{At}$ satisfies the two conditions

$$
\begin{align*}
(1) \quad & \frac{d}{dt} \Phi(t) = A\Phi(t), \\
(2) \quad & \Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
$$
Putzer Matrix Exponential Formula for $2 \times 2$ Matrices

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} (A - \lambda_1 I)$$  
\text{A is } 2 \times 2, \lambda_1 \neq \lambda_2 \text{ real.}

$$e^{At} = e^{\lambda_1 t} I + te^{\lambda_1 t} (A - \lambda_1 I)$$  
\text{A is } 2 \times 2, \lambda_1 = \lambda_2 \text{ real.}

$$e^{At} = e^{at} \cos bt I + \frac{e^{at} \sin bt}{b} (A - aI)$$  
\text{A is } 2 \times 2, \lambda_1 = \bar{\lambda}_2 = a + ib, \quad b > 0.$
How to Remember Putzer’s 2 × 2 Formula

The expressions

\[ e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I), \]

(1)

\[ r_1(t) = e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \]

are enough to generate all three formulas. Fraction \( r_2 \) is the \( d/d\lambda \)-Newton difference quotient for \( r_1 \). Then \( r_2 \) limits as \( \lambda_2 \to \lambda_1 \) to the \( d/d\lambda \)-derivative \( te^{\lambda_1 t} \). Therefore, the formula includes the case \( \lambda_1 = \lambda_2 \) by limiting. If \( \lambda_1 = \overline{\lambda_2} = a + ib \) with \( b > 0 \), then the fraction \( r_2 \) is already real, because it has for \( z = e^{\lambda_1 t} \) and \( w = \lambda_1 \) the form

\[ r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}. \]

Taking real parts of expression (1) gives the complex case formula.
**Variation of Parameters**

**Theorem 1 (Variation of Parameters for Linear Systems)**
Let $A$ be a constant $n \times n$ matrix and $\vec{F}(t)$ a continuous function near $t = t_0$. The unique solution $\vec{x}(t)$ of the matrix initial value problem

$$\vec{x}'(t) = A\vec{x}(t) + \vec{F}(t), \quad \vec{x}(t_0) = \vec{x}_0,$$

is given by the **variation of parameters formula**

$$\vec{x}(t) = e^{At}\vec{x}_0 + e^{At} \int_{t_0}^{t} e^{-rA}\vec{F}(r)dr.$$
Undetermined Coefficients

Theorem 2 (Polynomial Solutions)

Let $f(t)$ be a polynomial of degree $k$. Assume $A$ is an $n \times n$ constant invertible matrix. Then $\vec{u}' = A\vec{u} + f(t)\vec{c}$ has a polynomial solution $\vec{u}(t) = \sum_{j=0}^{k} \vec{c}_j t^{j/j!}$ of degree $k$ with vector coefficients $\{\vec{c}_j\}$ given by the relations

$$\vec{c}_j = -\sum_{i=j}^{k} f^{(i)}(0) A^{j-i-1} \vec{c}, \quad 0 \leq j \leq k.$$

Changes from $n$th Order Undetermined Coefficients. The $n$th order theory using Rule I and Rule II is replaced by

Systems Rule for Undetermined Coefficients. Assume $\frac{d}{dt} \vec{u} = A\vec{u} + \vec{F}(t)$. Extract all Euler atoms from $\vec{F}$, $\vec{F}'$, ... Don’t replace atoms by groups (Rule II). Instead, extend each existing group (Rule I) by adding $m - 1$ higher power terms $x^k$ (base atom) to the group, where $m$ is the multiplicity of the root for the base atom in the characteristic equation $|A - rI| = 0$. The trial solution is a linear combination of the final atom list with vector coefficients.