Systems of Differential Equations
The Eigenanalysis Method

- First Order $2 \times 2$ Systems $\mathbf{x}' = A\mathbf{x}$
- First Order $3 \times 3$ Systems $\mathbf{x}' = A\mathbf{x}$
- Second Order $3 \times 3$ Systems $\mathbf{x}'' = A\mathbf{x}$
- Vector-Matrix Form of the Solution of $\mathbf{x}' = A\mathbf{x}$
- Four Methods for Solving a System $\mathbf{x}' = A\mathbf{x}$
The Eigenanalysis Method for First Order $2 \times 2$ Systems

Suppose that $A$ is $2 \times 2$ real and has eigenpairs

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2),$$

with $\vec{v}_1$, $\vec{v}_2$ independent. The eigenvalues $\lambda_1$, $\lambda_2$ can be both real. Also, they can be a complex conjugate pair $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$.

**Theorem 1 (Eigenanalysis Method)**

The general solution of $\ddot{x} = A\dot{x}$ is

$$\dot{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$
Solving $2 \times 2$ Systems $\vec{x}' = A\vec{x}$ with Complex Eigenvalues

If the eigenvalues are complex conjugates, then the real part $\vec{w}_1$ and the imaginary part $\vec{w}_2$ of the solution $e^{\lambda_1 t}\vec{v}_1$ are independent solutions of the differential equation. Then the general solution in real form is given by the relation

$$\vec{x}(t) = c_1 \vec{w}_1(t) + c_2 \vec{w}_2(t).$$
The Eigenanalysis Method for First Order $3 \times 3$ Systems

Suppose that $A$ is $3 \times 3$ real and has eigenpairs

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2), \quad (\lambda_3, \vec{v}_3),$$

with $\vec{v}_1, \vec{v}_2, \vec{v}_3$ independent. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ can be all real. Also, there can be one real eigenvalue $\lambda_3$ and a complex conjugate pair of eigenvalues $\lambda_1 = \lambda_2 = a + ib$ with $b > 0$.

**Theorem 2 (Eigenanalysis Method)**
The general solution of $\vec{x}' = A\vec{x}$ with $3 \times 3$ real $A$ can be written as

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3.$$
Solving $3 \times 3$ Systems $\ddot{\mathbf{x}} = A\mathbf{x}$ with Complex Eigenvalues

If there are complex eigenvalues $\lambda_1 = \bar{\lambda}_2$, then the real general solution is expressed in terms of independent solutions

$$\mathbf{w}_1 = \text{Re}(e^{\lambda_1 t} \mathbf{v}_1), \quad \mathbf{w}_2 = \text{Im}(e^{\lambda_1 t} \mathbf{v}_1)$$

as the linear combination

$$\mathbf{x}(t) = c_1 \mathbf{w}_1(t) + c_2 \mathbf{w}_2(t) + c_3 e^{\lambda_3 t} \mathbf{v}_3.$$
Theorem 3 (Second Order Systems)

Let $A$ be real and $3 \times 3$ with three negative eigenvalues $\lambda_1 = -\omega_1^2$, $\lambda_2 = -\omega_2^2$, $\lambda_3 = -\omega_3^2$. Let the eigenpairs of $A$ be listed as

$$(\lambda_1, \vec{v}_1), \ (\lambda_2, \vec{v}_2), \ (\lambda_3, \vec{v}_3).$$

Then the general solution of the second order system $\ddot{\vec{x}}(t) = A\dot{\vec{x}}(t)$ is

$$\vec{x}(t) = \left( a_1 \cos \omega_1 t + b_1 \frac{\sin \omega_1 t}{\omega_1} \right) \vec{v}_1$$

$$+ \left( a_2 \cos \omega_2 t + b_2 \frac{\sin \omega_2 t}{\omega_2} \right) \vec{v}_2$$

$$+ \left( a_3 \cos \omega_3 t + b_3 \frac{\sin \omega_3 t}{\omega_3} \right) \vec{v}_3.$$
Vector-Matrix Form of the Solution of $\vec{x}' = A\vec{x}$

The solution of $\vec{x}' = A\vec{x}$ in the $3 \times 3$ case is written in vector-matrix form

$$\vec{x}(t) = \text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$ 

This formula is normally used when the eigenpairs are real.
Complex Eigenvalues for a $2 \times 2$ System

When there is a complex conjugate pair of eigenvalues $\lambda_1 = \lambda_2 = a + ib, b > 0$, then it is possible to extract a real solution $\vec{x}$ from the complex formula and report a real solution. The work can be organized more efficiently using the matrix product

$$\vec{x}(t) = e^{at} \text{aug}(\text{Re}(\vec{v}_1), \text{Im}(\vec{v}_1)) \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$
Complex Eigenvalues for a $3 \times 3$ System

When there is a complex conjugate pair of eigenvalues $\lambda_1 = \lambda_2 = a + ib$, $b > 0$, then a real solution $\vec{x}$ can be extracted from the complex formula to report a real solution. The work is organized using the matrix product

$$\vec{x}(t) = \text{aug}(\text{Re}(\vec{v}_1), \text{Im}(\vec{v}_1), \vec{v}_3) \begin{pmatrix} e^{at} \cos bt & e^{at} \sin bt & 0 \\ -e^{at} \sin bt & e^{at} \cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. $$
Four Methods for Solving a $2 \times 2$ System $\vec{u}' = A\vec{u}$

1. **First-order method.** If $A$ is diagonal, then use growth-decay methods. If $A$ is triangular, then use the linear integrating factor method.

2. **Cayley-Hamilton-Ziebur method.** If $A$ is not diagonal, and $a_{12} \neq 0$, then $u_1(t)$ is a linear combination of the atoms constructed from the roots $r$ of $\det(A - rI) = 0$. Solution $u_2(t)$ is found from the system by solving for $u_2$ in terms of $u_1$ and $u_1'$.

3. **Eigenanalysis method.** Assume $A$ has eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$ with $\vec{v}_1$, $\vec{v}_2$ independent. Then $\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$.

4. **Resolvent method.** In Laplace notation, $\vec{u}(t) = L^{-1}((sI - A)^{-1}\vec{u}(0))$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \text{adj}(C) / \det(C)$. Cramer’s Rule can replace the matrix inversion method.
Four Methods for Solving an $n \times n$ System $\vec{u}' = A\vec{u}$

1. **First-order method.** If $A$ is diagonal, then use growth-decay methods. If $A$ is triangular, then use the linear integrating factor method.

2. **Cayley-Hamilton-Ziebur method.** The solution $\vec{u}(t)$ is a linear combination of the atoms constructed from the roots $r$ of $\det(A - rI) = 0$,

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n.$$ 

To solve for the constant vectors $\vec{d}_j$, differentiate the formula $n - 1$ times, then use $A^k\vec{u}(t) = \vec{u}^{(k+1)}(t)$ and set $t = 0$, to obtain a system for $\vec{d}_1, \ldots, \vec{d}_n$.

3. **Eigenanalysis method.** Assume $A$ has eigenpairs $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ with $\vec{v}_1, \ldots, \vec{v}_n$ independent. Then $\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$.

4. **Resolvent method.** In Laplace notation, $\vec{u}(t) = L^{-1}((sI - A)^{-1}\vec{u}(0))$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \text{adj}(C)/\det(C)$. Cramer’s Rule can replace the matrix inversion method.