Systems of Second Order Differential Equations
Cayley-Hamilton-Ziebur

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Definition 1 (Characteristic Equation)
Given a square matrix $A$, the characteristic equation of $A$ is the polynomial equation

$$\det(A - \lambda I) = 0.$$ 

The determinant $\det(A - \lambda I)$ is formed by subtracting $\lambda$ from the diagonal of $A$. The polynomial $p(x) = \det(A - xI)$ is called the characteristic polynomial of matrix $A$.

- If $A$ is $2 \times 2$, then $p(x)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(x)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.
Characteristic Equation Examples

Create $\det(A - xI)$ by subtracting $x$ from the diagonal of $A$.
Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 \\ 0 & 4 - x \end{vmatrix} = (2 - x)(4 - x)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 & 4 \\ 0 & 5 - x & 6 \\ 0 & 0 & 7 - x \end{vmatrix} = (2-x)(5-x)(7-x)$$
Theorem 1 (Cayley-Hamilton)
A square matrix \( A \) satisfies its own characteristic equation.

If \( p(x) = (-x)^n + a_{n-1}(-x)^{n-1} + \cdots + a_0 \), then the result is the equation
\[
(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,
\]
where \( I \) is the \( n \times n \) identity matrix and \( 0 \) is the \( n \times n \) zero matrix.

The 2 \( \times \) 2 Case

Then \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and for \( a_1 = \text{trace}(A), \ a_0 = \text{det}(A) \) we have \( p(x) = x^2 + a_1(-x) + a_0 \). The Cayley-Hamilton theorem says
\[
A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Cayley-Hamilton Example

Assume

\[ A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix} \]

Then

\[ p(x) = \begin{vmatrix} 2 - x & 3 & 4 \\ 0 & 5 - x & 6 \\ 0 & 0 & 7 - x \end{vmatrix} = (2 - x)(5 - x)(7 - x) \]

and the Cayley-Hamilton Theorem says that

\[ (2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \]
Euler’s Substitution and the Characteristic Equation

Definition. Euler’s Substitution for the second order equation $\vec{u}'' = A\vec{u}$ is

$$\vec{u} = \vec{v}e^{rt}.$$

The symbol $r$ is a real or complex constant and symbol $\vec{v}$ is a constant vector.

Theorem 2 (Euler Solution Equation from Euler’s Substitution)
Euler’s substitution applied to $\vec{u}'' = A\vec{u}$ leads directly to the equation

$$|A - r^2 I| = 0.$$

This is perhaps the premier method for remembering the characteristic equation for the second order vector-matrix equation $\vec{u}'' = A\vec{u}$.

Proof: Substitute $\vec{u} = \vec{v}e^{rt}$ into $\vec{u}'' = A\vec{u}$ to obtain $r^2 e^{rt} \vec{v} = A\vec{v}e^{rt}$. Cancel the exponential, then $r^2 \vec{v} = A\vec{v}$. Re-arrange to the homogeneous system $(A - r^2 I) \vec{v} = \vec{0}$. This homogeneous linear algebraic equation has a nonzero solution $\vec{v}$ if and only if the determinant of coefficients vanishes: $|A - r^2 I| = 0$. 

Theorem 3 (Cayley-Hamilton-Ziebur Structure Theorem for $\ddot{\mathbf{u}} = A\dot{\mathbf{u}}$)

The solution $\ddot{\mathbf{u}}(t)$ of second order equation $\ddot{\mathbf{u}}(t) = A\dot{\mathbf{u}}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2I) = 0$.

The equation $|A - r^2I| = 0$ is formed by substitution of $\lambda = r^2$ into the eigenanalysis characteristic equation of $A$.

In symbols, the structure theorem says

$$\ddot{\mathbf{u}} = \ddot{\mathbf{d}}_1 A_1 + \cdots + \ddot{\mathbf{d}}_k A_k,$$

where $A_1, \ldots, A_k$ are Euler solution atoms corresponding to the roots $r$ of the determining equation $|A - r^2I| = 0$. Therefore, all vectors in the relation have real entries. However, only $2n$ entries of vectors $\ddot{\mathbf{d}}_1, \ldots, \ddot{\mathbf{d}}_k$ are arbitrary constants, the remaining entries being dependent on them.
Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case when \( A \) is \( 2 \times 2 \) (\( n = 2 \)), because the proof details are similar in higher dimensions. Expand \( |A - xI| = 0 \) to find the characteristic equation \( x^2 + cx + d = 0 \), for some constants \( c, d \). The Cayley-Hamilton theorem says that \( A^2 + cA + dI = 0 \). Let \( \vec{u} \) be a solution of \( \vec{u}''(t) = A\vec{u}(t) \). Multiply the Cayley-Hamilton identity by vector \( \vec{u} \) and simplify to obtain

\[
A^2\vec{u} + cA\vec{u} + d\vec{u} = \vec{0}.
\]

Using equation \( \vec{u}''(t) = A\vec{u}(t) \) backwards, we compute \( A^2\vec{u} = A\vec{u}'' = \vec{u}''' \). Replace the terms of the displayed equation to obtain the relation

\[
\vec{u}''' + c\vec{u}'' + d\vec{u} = \vec{0}.
\]

Each component \( y \) of vector \( \vec{u} \) then satisfies the 4th order linear homogeneous equation \( y^{(4)} + cy^{(2)} + dy = 0 \), which has characteristic equation \( r^4 + cr^2 + d = 0 \). This equation is the expansion of determinant equation \( |A - r^2I| = 0 \). Therefore \( y \) is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that \( \vec{u} \) is a vector linear combination of the Euler solution atoms so identified. This completes the proof.
A 2 × 2 Illustration

Solve the system $\ddot{\mathbf{u}} = A\mathbf{u}$, $A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$, which is a spring-mass system with $k_1 = 100$, $k_2 = 50$, $m_1 = 2$, $m_2 = 1$.

Solution: The eigenvalues of $A$ are $\lambda = -25$ and $-100$. Then the determining equation $|A - r^2 I| = 0$ has complex roots $r = \pm 5i$ and $\pm 10i$ with corresponding Euler solution atoms $\cos(4t), \sin(5t), \cos(10t), \sin(10t)$. The eigenpairs of $A$ are

\[
\left( -25, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right), \quad \left( -100, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).
\]

Then $\mathbf{u}$ is a vector linear combination of the Euler solution atoms

\[
\mathbf{u}(t) = d_1 \cos(5t) + d_2 \sin(5t) + d_3 \cos(10t) + d_4 \sin(10t).
\]
How to Find $\vec{d}_1$ to $\vec{d}_4$

Substitute the formula

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)$$

into $\vec{u}'' = A\vec{u}$, then solve for the unknown vectors $\vec{d}_j$, $j = 1, 2, 3, 4$, by equating coefficients of Euler solution atoms matching left and right:

$$A\vec{d}_1 = -25\vec{d}_1, \quad A\vec{d}_2 = -25\vec{d}_2, \quad A\vec{d}_3 = -100\vec{d}_3, \quad A\vec{d}_4 = -100\vec{d}_4.$$ 

These eigenpair relationships imply formulas involving the eigenvectors of $A$. We get, for some constants $a_1, a_2, b_1, b_2$,

$$\vec{d}_1 = a_1 \left( \begin{array}{c} 1 \\ 2 \end{array} \right), \quad \vec{d}_2 = b_1 \left( \begin{array}{c} 1 \\ 2 \end{array} \right), \quad \vec{d}_3 = a_2 \left( \begin{array}{c} 1 \\ -1 \end{array} \right), \quad \vec{d}_4 = b_2 \left( \begin{array}{c} 1 \\ -1 \end{array} \right).$$
Summary for the $2 \times 2$ Illustration

\[
\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)
\]

\[
\vec{u}(t) = (a_1 \cos(5t) + b_1 \sin(5t)) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos(10t) + b_2 \sin(10t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]