

Laplace Theory Examples

- **Harmonic oscillator**
- **s -Differentiation Rule**
- **First shifting rule**
- **Trigonometric formulas**
- **Exponentials**
- **Hyperbolic functions**
- **s -Differentiation Rule**
- **First Shifting Rule I and II**
- **Damped oscillator**
- **Second Shifting Rule I and II**

1 Example (Harmonic oscillator) Solve the initial value problem $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$ by Laplace's method.

Solution:

The solution is $x(t) = \sin t$. The details:

$$L(x'') + L(x) = L(0)$$

$$sL(x') - x'(0) + L(x) = 0$$

$$s[sL(x) - x(0)] - x'(0) + L(x) = 0$$

$$(s^2 + 1)L(x) = 1$$

$$L(x) = \frac{1}{s^2 + 1}$$
$$= L(\sin t)$$

$$x(t) = \sin t$$

Apply L across the equation.

Parts rule.

Parts rule again.

Use $x(0) = 0$, $x'(0) = 1$.

Isolate $L(x)$ left.

Backward Laplace table.

Lerch's cancelation law.

2 Example (*s*-Differentiation Rule) Show the steps for $L(t^2 e^{5t}) = \frac{2}{(s - 5)^3}$.

Solution:

$$\begin{aligned}L(t^2 e^{5t}) &= \left(-\frac{d}{ds}\right) \left(-\frac{d}{ds}\right) L(e^{5t}) \\&= (-1)^2 \frac{d}{ds} \frac{d}{ds} \left(\frac{1}{s - 5}\right) \\&= \frac{d}{ds} \left(\frac{-1}{(s - 5)^2}\right) \\&= \frac{2}{(s - 5)^3}\end{aligned}$$

Use *s*-differentiation.

Forward Laplace table.

Calculus power rule.

Identity verified.

3 Example (First shifting rule) Show the steps for $L(t^2 e^{-3t}) = \frac{2}{(s + 3)^3}$.

Solution:

$$\begin{aligned} L(t^2 e^{-3t}) &= L(t^2) \Big|_{s \rightarrow s - (-3)} \\ &= \left(\frac{2}{s^2 + 1} \right) \Big|_{s \rightarrow s - (-3)} \\ &= \frac{2}{(s + 3)^3} \end{aligned}$$

First shifting rule.

Forward Laplace table.

Identity verified.

4 Example (Trigonometric formulas) Show the steps used to obtain these Laplace identities:

$$(a) L(t \cos bt) = \frac{s^2 - b^2}{(s^2 + b^2)^2}$$

$$(c) L(t^2 \cos bt) = \frac{2(s^3 - 3sb^2)}{(s^2 + b^2)^3}$$

$$(b) L(t \sin bt) = \frac{2sb}{(s^2 + b^2)^2}$$

$$(d) L(t^2 \sin bt) = \frac{6s^2b - b^3}{(s^2 + b^2)^3}$$

Solution: The details for (b) and (d) are left as exercises.

The details for (a):

$$\begin{aligned} L(t \cos bt) &= -(d/ds)L(\cos bt) \\ &= -\frac{d}{ds} \left(\frac{s}{s^2 + b^2} \right) \\ &= \frac{s^2 - b^2}{(s^2 + b^2)^2} \end{aligned}$$

Use s -differentiation.

Forward Laplace table.

Calculus quotient rule.

The details for (c):

$$\begin{aligned} L(t^2 \cos bt) &= -(d/ds)L((-t) \cos bt) \\ &= \frac{d}{ds} \left(-\frac{s^2 - b^2}{(s^2 + b^2)^2} \right) \\ &= \frac{2s^3 - 6sb^2}{(s^2 + b^2)^3} \end{aligned}$$

Use s -differentiation.

Result of (a).

Calculus quotient rule.

5 Example (Exponentials) Show the steps used to obtain these Laplace identities:

$$(a) L(e^{at} \cos bt) = \frac{s - a}{(s - a)^2 + b^2} \quad (c) L(te^{at} \cos bt) = \frac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2}$$

$$(b) L(e^{at} \sin bt) = \frac{b}{(s - a)^2 + b^2} \quad (d) L(te^{at} \sin bt) = \frac{2b(s - a)}{((s - a)^2 + b^2)^2}$$

Solution: Left as exercises are (a), (b) and (d).

Details for (c):

$$\begin{aligned} L(te^{at} \cos bt) &= L(t \cos bt)|_{s \rightarrow s-a} \\ &= \left(-\frac{d}{ds} L(\cos bt) \right) \Big|_{s \rightarrow s-a} \\ &= \left(-\frac{d}{ds} \left(\frac{s}{s^2 + b^2} \right) \right) \Big|_{s \rightarrow s-a} \\ &= \left(\frac{s^2 - b^2}{(s^2 + b^2)^2} \right) \Big|_{s \rightarrow s-a} \\ &= \frac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2} \end{aligned}$$

First shifting rule.

Use s -differentiation.

Forward Laplace table.

Calculus quotient rule.

Verified (c).

6 Example (Hyperbolic functions) Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

$$(a) L(\cosh at) = \frac{s}{s^2 - a^2}$$

$$(b) L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$(c) L(t \cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2}$$

$$(d) L(t \sinh at) = \frac{2as}{(s^2 - a^2)^2}$$

Solution: Left as exercises are (b) and (c).

The details for (a):

$$\begin{aligned} L(\cosh at) &= \frac{1}{2}(L(e^{at}) + L(e^{-at})) \\ &= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) \\ &= \frac{s}{s^2 - a^2} \end{aligned}$$

Definition and linearity.

Forward Laplace table.

Identity (a) verified.

The details for (d):

$$\begin{aligned} L(t \sinh at) &= -\frac{d}{ds} \left(\frac{a}{s^2 - a^2} \right) \\ &= \frac{a(2s)}{(s^2 - a^2)^2} \end{aligned}$$

Apply the s -differentiation rule.

Calculus power rule; (d) verified.

7 Example (s -Differentiation Rule) Solve $L(f(t)) = \frac{2s}{(s^2 + 1)^2}$ for $f(t)$.

Solution: The solution is $f(t) = t \sin t$.

The details:

$$\begin{aligned}L(f(t)) &= \frac{2s}{(s^2 + 1)^2} \\&= -\frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) \\&= -\frac{d}{ds} (L(\sin t)) \\&= L(t \sin t)\end{aligned}$$

$$f(t) = t \sin t$$

Calculus power rule $(u^n)' = nu^{n-1}u'$.

Backward Laplace table.

Apply the s -differentiation rule.

Lerch's cancelation law.

8 Example (First Shifting Rule I) Solve $L(f(t)) = \frac{s + 7}{s^2 + 4s + 8}$ for $f(t)$.

Solution:

The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t)$. The details:

$$L(f(t)) = \frac{s + 7}{(s + 2)^2 + 4}$$

$$= \frac{u + 5}{u^2 + 4}$$

$$= \frac{u}{u^2 + 4} + \frac{5}{2} \frac{2}{u^2 + 4}$$

$$= \frac{s}{s^2 + 4} + \frac{5}{2} \frac{2}{s^2 + 4} \Big|_{s \rightarrow u \equiv s+2}$$

$$= L\left(\cos 2t + \frac{5}{2} \sin 2t\right) \Big|_{s \rightarrow u \equiv s+2}$$

$$= L(e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t))$$

$$f(t) = e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t)$$

Complete the square.

Substitute $u = s + 2$.

Split into table entries.

Shifting rule preparation.

Backward Laplace table.

First shifting rule.

LeRch's cancelation law.

9 Example (First Shifting Rule II) Solve $L(f(t)) = \frac{s + 2}{s^2 + 2s + 2}$ for $f(t)$.

Solution:

The answer is $f(t) = e^{-t} \cos t + e^{-t} \sin t$. The details:

$$L(f(t)) = \frac{s + 2}{s^2 + 2s + 2}$$

$$= \frac{s + 2}{(s + 1)^2 + 1}$$

$$= \frac{u + 1}{u^2 + 1}$$

$$= \frac{u}{u^2 + 1} + \frac{1}{u^2 + 1}$$

$$= L(\cos t) + L(\sin t)|_{s \rightarrow u \equiv s+1}$$

$$= L(e^{-t} \cos t) + L(e^{-t} \sin t)$$

$$f(t) = e^{-t} \cos t + e^{-t} \sin t$$

Signal for this method: the denominator has complex roots.

Complete the square, denominator.

Substitute $u = s + 1$.

Split into Laplace table entries.

Backward Laplace table.

First shift rule.

Invoke Lerch's cancelation law.

10 Example (Damped oscillator) Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.

Solution:

The solution is $x(t) = e^{-t} \cos t$. The details:

$$L(x'') + 2L(x') + 2L(x) = L(0)$$

$$sL(x') - x'(0) + 2L(x') + 2L(x) = 0$$

$$s[sL(x) - x(0)] - x'(0) + 2[L(x) - x(0)] + 2L(x) = 0$$

$$(s^2 + 2s + 2)L(x) = 1 + s$$

$$L(x) = \frac{s + 1}{s^2 + 2s + 2}$$

$$= \frac{s + 1}{(s + 1)^2 + 1}$$

$$= L(\cos t)|_{s \rightarrow s+1}$$

$$= L(e^{-t} \cos t)$$

$$x(t) = e^{-t} \cos t$$

Apply L across the equation.

The parts rule on x' .

The parts rule on x .

Use $x(0) = 1$, $x'(0) = -1$.

Divide to isolate $L(x)$.

Complete the square in the denominator.

Forward Laplace table.

First shifting rule.

Lerch's cancelation law.

11 Example (Second Shifting Rule I) Show the steps for

$$L(\sin t u(t - \pi)) = \frac{-e^{-\pi s}}{s^2 + 1}, \quad u = \text{unit step.}$$

Solution:

The second shifting rule is applied as follows.

$$\begin{aligned} \text{LHS} &= L(\sin t u(t - \pi)) \\ &= L(g(t)u(t - a)) \\ &= e^{-as} L(g(t + a)) \\ &= e^{-\pi s} L(\sin(t + \pi)) \\ &= e^{-\pi s} L(-\sin t) \end{aligned}$$

$$= e^{-\pi s} \frac{-1}{s^2 + 1}$$

$$= \text{RHS}$$

Left side of the identity.

Choose $g(t) = \sin t$, $a = \pi$.

Second form, second shifting theorem.

Substitute $a = \pi$.

Sum rule $\sin(a + b) = \sin a \cos b + \sin b \cos a$ plus $\sin \pi = 0$, $\cos \pi = -1$.

Forward Laplace table.

Identity verified.

12 Example (Second Shifting Rule II) Solve for $f(t)$:

$$L(f(t)) = e^{-3s} \frac{s + 1}{s^2 + 2s + 2}$$

Solution:

The answer is $f(t) = e^{3-t} \cos(t - 3)u(t - 3)$, $u =$ unit step. The details:

$$L(f(t)) = e^{-3s} \frac{s + 1}{(s + 1)^2 + 1}$$

$$= e^{-3u+3} \frac{u}{u^2 + 1}$$

$$= e^{-3u+3} (L(\cos t))|_{s \rightarrow u \equiv s+1}$$

$$= e^3 (e^{-3u} L(\cos t))|_{s \rightarrow u \equiv s+1}$$

$$= e^3 (L(\cos(t - 3)u(t - 3)))|_{s \rightarrow u \equiv s+1}$$

$$= e^3 L(e^{-t} \cos(t - 3)u(t - 3))$$

$$f(t) = e^{3-t} \cos(t - 3)u(t - 3)$$

Complete the square.

Substitute $u = s + 1$.

Forward Laplace table.

Regroup factor e^{-3u} .

Second shifting rule.

First shifting rule.

Lerch's cancelation law.