Introduction to Linear Systems

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Linear Nonhomogeneous System

Given numbers $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$, consider the **nonhomogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$

(1)
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (1). Constants $b_1, \ldots, b_m$ are collectively referenced as the **right hand side**, **right side** or **RHS**.

Linear Homogeneous System

Given numbers $a_{11}, \ldots, a_{mn}$ consider the **homogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$

(2)
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.
\end{aligned}
$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (2).

The Three Possibilities

Solutions of general linear system (1) may be classified into exactly **three possibilities**:

1. No solution.
2. Infinitely many solutions.
3. A unique solution.

Examples ─────────────────────────────────────

A **solution** (x, y) of a 2×2 system is a pair of values that simultaneously satisfy both equations. Consider the following three systems:

1 **No Solution**

$$\begin{cases} x & - & y & = & 0, \\ & & 0 & = & 1. \end{cases}$$

2 **Infinitely Many Solutions**

$$\begin{cases} x & - & 2y & = & 0, \\ & & 0 & = & 0. \end{cases}$$

3 **Unique Solution**

$$\begin{cases} 3x & + & 2y & = & 1, \\ x & - & y & = & 2. \end{cases}$$

Three Possibilities Explained ─────────────────────────

- **Signal equation** $0 = 1$, a false equation, implies system **1** has **No Solution**. Geometrically realized by two parallel lines.

- System **2** represents 2 identical lines. It has **infinitely many solutions**, one solution (x, y) for each point on the line $x - 2y = 0$. Analytic geometry writes the solutions for $-\infty < t_1 < \infty$ as the *parametric equations*

$$\begin{cases} x & = & 2t_1, \\ y & = & t_1. \end{cases}$$

- System **3** represents 2 non-parallel lines. It has a **unique solution** $x = 1, y = -1$.

Toolkit: Combo, Swap, Mult

The toolkit rules *neither create nor destroy solutions* of the original system.

Swap   Two equations can be interchanged without changing the solution set.

Mult   An equation can be multiplied by $m \neq 0$ without changing the solution set.

Combo   A multiple of one equation can be added to a different equation without changing the solution set.

Reversible Rules

The `mult` and `combo` rules replace an existing equation by a new one, whereas `swap` replaces two equations. The three operations are **reversible**:

- The `swap` rule is reversed by repeating it.

- The `mult` rule is reversed by repeating it with multiplier $1/m$.

- The `combo` rule is reversed by repeating it with c replaced by $-c$.

Reduced Echelon Systems _____

- The **leading term** in an equation is the first nonzero term in variable list order. The **leading coefficient** is the nonzero number multiplying the variable in the leading term.

- A **lead variable** is a variable that appears exactly once in the system of equations, in a leading term with leading coefficient one.

- A system of linear algebraic equations in which each nonzero equation has a **lead variable** is called a **reduced echelon system**. The conventions:

  - Within an equation, variables must appear in variable list order.
  - Equations with lead variables are listed in variable list order.
  - Following them are any zero equations.
  - A nonzero equation with no variables is not allowed.
    Called a **signal equation**, it looks like **0=1**.

- A **free variable** in a reduced echelon system is any variable that is not a lead variable. A variable that does not appear is a free variable. Otherwise, detection of a free variable must be from a triangular system in which all lead variables are identified.

Recognition of Reduced Echelon Systems ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

A linear system (1) is recognized as possible to convert into a reduced echelon system provided the leading term in each equation has its variable missing from all other equations. Conversion is made by `mult` and `swap` toolkit rules.

Form of a Reduced Echelon System ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

A reduced echelon system can be written in the special form

(3)
$$
\begin{aligned}
x_{i_1} + E_{11}x_{j_1} + \cdots + E_{1k}x_{j_k} &= D_1, \\
x_{i_2} + E_{21}x_{j_1} + \cdots + E_{2k}x_{j_k} &= D_2, \\
&\ \ \vdots \\
x_{i_m} + E_{m1}x_{j_1} + \cdots + E_{mk}x_{j_k} &= D_m.
\end{aligned}
$$

Symbols Defined ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

- The numbers $E_{11}, \ldots, E_{mk}$ and $D_1, \ldots, D_m$ are *known constants*.

- Variables $x_{i_1}, \ldots, x_{i_m}$ are the lead variables.

- The remaining variables $x_{j_1}, \ldots, x_{i_k}$ are the free variables.

Writing the Solution when Free Variables Exist

Assume variable list order x, y, z, w, u, v for the **reduced echelon system** below. Boxed variables are lead variables and the remaining are free variables.

(4)
$$\begin{aligned}
\boxed{x} + 4w + u + v &= 1, \\
\boxed{y} - u + v &= 2, \\
\boxed{z} - w + 2u - v &= 0.
\end{aligned}$$

Last Frame Algorithm

If the reduced echelon system has zero free variables, then the unique solution is already displayed. Otherwise, there is at least one free variable, and then the 2–step algorithm below applies to write out the general solution.

1. Set the free variables equal to *invented symbols $t_1, \ldots, t_k$*. Each symbol can assume values from $-\infty$ to ∞.

2. Solve equations (4) for the leading variables and then back-substitute the free variables to obtain a **standard general solution**.

From Reduced Echelon System To General Solution ——————————

Assume variable list order x, y, z, w, u, v in the reduced echelon system

(5)
$$\begin{aligned}
\boxed{x} + 4w + u + v &= 1, \\
\boxed{y} - u + v &= 2, \\
\boxed{z} - w + 2u - v &= 0.
\end{aligned}$$

The boxed **lead variables** in (5) are x, y, z and the **free variables** are w, u, v. Assign invented symbols t_1, t_2, t_3 to the free variables and back-substitute in (5) to obtain a **standard general solution**

$$\begin{cases}
x &= 1 - 4t_1 - t_2 - t_3, \\
y &= 2 + t_2 - t_3 \\
z &= t_1 - 2t_2 + t_3, \\
w &= t_1, \\
u &= t_2, \\
v &= t_3.
\end{cases}$$

Elimination Algorithm

The algorithm employs at each algebraic step one of the **three rules** in the Toolkit **mult**, **swap** and **combo**.

- The objective of each algebraic step is to **increase the number of lead variables**. The process stops when no more lead variables can be found, in which case the last system of equations is a **reduced echelon system**. It may also stop when a signal equation is found. Otherwise, equations with lead variables migrate to the top, in variable list order dictated by the lead variable, and equations with no variables are swapped to the end. Within each equation, variables appear in variable list order, left-to-right.

- Reversibility of the algebraic steps means that no solutions are created or destroyed: the original system and all systems in the intermediate steps have *exactly the same solutions*.

- The final reduced echelon system has a standard **general solution**. This expression is either the unique solution of the system, or else in the case of infinitely many solutions, it is a parametric solution using invented symbols $t_1, \ldots, t_k$.

Theorem 1 (Method of Elimination)
Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the three toolkit rules **swap**, **mult** and **combo**.

Documenting the 3 Rules ──

Throughout, symbols s and t stand for source and target equations. Symbols c and m stand for constant and multiplier. The multiplier must be nonzero. The symbol R used in textbooks abbreviates **Row**, which corresponds exactly to **equation**.

Swap(s,t)   Interchange equation/row s and equation/row t.
        Textbooks: SWAP R_s, R_t
        Blackboard: swap(s,t)

Mult(t,m)   Multiply equation/row t by $m \neq 0$.
        Textbooks: $(m)R_t$
        Blackboard: mult(t,m)

Combo(s,t,c)   Multiply equation/row s by c and then add it to a different equation/row t.
        Textbooks: $R_t = (c)R_s + R_t$
        Blackboard: combo(s,t,c)

From Equations to Matrices

The system $\begin{cases} 3x + 4y = 1 \\ 5x - 6y = 2 \end{cases}$ can be represented as the augmented matrix $C = \left(\begin{array}{cc|c} 3 & 4 & 1 \\ 5 & -6 & 2 \end{array} \right)$. The process is automated for any number of variables, by observing that each column of C is a partial derivative, e.g., column 1 of C is the partial on x, column 2 of C the partial on y. The rightmost column of C contains the RHS of each equation.

From Matrices to Equations

The variable names x, y are written above columns 1, 2 of the augmented matrix, as below.

$$\begin{matrix} \mathbf{x} & \mathbf{y} & \mathbf{=} \end{matrix}$$
$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 5 & -6 & 2 \end{array} \right)$$

The scalar system is reconstructed from the augmented matrix by taking dot products, of the symbols on the top row against the rows of the matrix. Each dot product answer is followed by an equal sign and then the number in the third column.

The Three Rules and `Maple`

Newer versions of `maple` use the `LinearAlgebra` package, and a single function `RowOperation()` to accomplish the same thing done by three functions `addrow()`, `mulrow()`, `swaprow()` in the older `linalg` package. A conversion table appears below.

Hand-written	maple linalg	maple LinearAlgebra
swap(s,t)	swaprow(A,s,t)	RowOperation(A,[t,s])
mult(t,c)	mulrow(A,t,c)	RowOperation(A,t,c)
combo(s,t,c)	addrow(A,s,t,c)	RowOperation(A,[t,s],c)

Handwritten to Maple Code Conversion

Ref: Utah Maple Tutorial, course web site.

```
swap:=(A,s,t)->linalg[swaprow](A,s,t):
mult:=(A,t,m)->linalg[mulrow](A,t,m):
combo:=(A,s,t,c)->linalg[addrow](A,s,t,c):
C1 := Matrix([[3,2,2],[3,1,2],[1,1,1]]);
C2:=combo(C1,1,2,-1);
C3:=combo(C2,1,3,-1/3);
```

RREF

The reduced row-echelon form of a matrix, or **rref**, is specified by the following requirements. The matrix can be A for $A\vec{x} = \vec{0}$ or the augmented matrix $\langle A|\vec{b}\rangle$ for $A\vec{x} = \vec{b}$. This means a signal equation $0 = 1$ is possible, but it depends on interpretation.

- Zero rows appear last.

- Each nonzero row has first element 1, called a **leading one**. The column in which the leading one appears, called a **pivot column**, has all other entries equal to zero.

- The pivot columns appear as *consecutive* initial columns of the identity matrix I. Trailing columns of I might be absent.

RREF Illustration

The matrix (6) below is a typical **rref** which satisfies the preceding properties. Displayed secondly is the reduced echelon system (7) in the variables $x_1, \ldots, x_8$ represented by the augmented matrix (6).

(6)
$$\left(\begin{array}{cccccccc|c} 1 & 2 & 0 & 3 & 4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 1 & 7 & 8 & 0 & 9 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 11 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(7)
$$\begin{cases} x_1 + 2x_2 & +3x_4 +4x_5 & + 5x_7 & = 6 \\ x_3 +7x_4 +8x_5 & + 9x_7 & = 10 \\ x_6 +11x_7 & = 12 \\ x_8 & = 13 \\ 0 & = 0 \\ 0 & = 0 \\ 0 & = 0 \end{cases}$$

Matrix (6) is an **rref**, because it passes the two checkpoints. The pivot columns 1, 3, 6, 8 appear as the initial 4 columns of the 7×7 identity matrix I, in natural order; the trailing 3 columns of I are absent.

Linear System Frame Sequence

- A sequence of swap, multiply and combination steps applied to a system of equations is called a **frame sequence**. Each step (a frame) consists of exactly one operation.

- The **First Frame** is the original system and the **Last Frame** is the reduced echelon system.

- Frames are documented by the acronyms `swap, combo, mult`. Arithmetic detail is suppressed: only results appear.

- The *viewpoint* is that a camera is pointed at the workspace of an assistant who writes the mathematics, and after the completion of each step, a photo is taken. The filmstrip of photos is the frame sequence. The sequence is related to a video of the events, because it is filmstrip of certain video frames.

Matrix Frame Sequences

The same terminology applies for systems $A\vec{x} = \vec{b}$ represented by an augmented matrix $C = \mathbf{aug}(A, \vec{b})$. The First Frame is C and the Last Frame is $\mathbf{rref}(C)$.

A Matrix Frame Sequence Illustration ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯
Steps in a frame sequence can be documented by the notation

$$\texttt{swap(s,t), mult(t,m), combo(s,t,c),}$$

each written next to the target row.

During the sequence, initial columns of the identity, called **pivot columns**, are created as steps toward the **rref**, using the three operations `mult, swap, combo`.

Trailing columns of the identity might not appear. Required is that pivot columns occur as consecutive initial columns of the identity matrix.

Frame 1: $\begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 1 & 4 & -1 & 0 & | & 2 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ Original augmented matrix.

Frame 2: $\begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 2 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `combo(1,2,-1)`
Pivot column 1 completed.

Frame 3: $\begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 2 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `swap(2,3)`

Frame 4: $\begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & -2 & 0 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `combo(2,3,-2)`

Frame 5: $\begin{pmatrix} 1 & 0 & -3 & 0 & | & -1 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & -2 & 0 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `combo(2,1,-2)`
Pivot column 2 completed.

Frame 6: $\begin{pmatrix} 1 & 0 & -3 & 0 & | & -1 \\ 0 & 1 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `mult(3,-1/2)`

Frame 7: $\begin{pmatrix} 1 & 0 & -3 & 0 & | & -1 \\ 0 & 1 & 0 & 0 & | & 1/2 \\ 0 & 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `combo(3,2,-1)`

Last Frame: $\begin{pmatrix} 1 & 0 & 0 & 0 & | & 1/2 \\ 0 & 1 & 0 & 0 & | & 1/2 \\ 0 & 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ `combo(3,1,3)`
Pivot column 3 completed.
rref found.

Definition 1 (Parametric Equations)

The terminology **parametric equations** refers to a set of equations of the form

(8)
$$
\begin{aligned}
x_1 &= d_1 + c_{11}t_1 + \cdots + c_{1k}t_k, \\
x_2 &= d_2 + c_{21}t_1 + \cdots + c_{2k}t_k, \\
&\vdots \\
x_n &= d_n + c_{n1}t_1 + \cdots + c_{nk}t_k.
\end{aligned}
$$

The numbers $d_1, \ldots, d_n, c_{11}, \ldots, c_{nk}$ are *known constants* and the variable names $t_1, \ldots, t_k$ are **parameters**. The symbols $t_1, \ldots, t_k$ are therefore allowed to take on any value from $-\infty$ to ∞.

Analytic Geometry

- In calculus courses, *parametric equations* are encountered as scalar equations of lines $(k = 1)$ and planes $(k = 2)$.

- If no symbols $t_1, t_2, \ldots$ appear, then the equations describe a *point*.

Definition 2 (General Solution)

A **general solution** of (1) is a set of parametric equations (8) plus two additional requirements:

(9)   Equations (8) satisfy (1) for all real values of $t_1, \ldots, t_k$.

(10)   Any solution of (1) can be obtained from (8) by specializing values of the parameter symbols $t_1, t_2, \ldots t_k$.