# Cayley-Hamilton Theorem

- Characteristic Equation
- Cayley-Hamilton Theorem
- An Example

## Characteristic Equation

**Definition 1 (Characteristic Equation)**
Given a square matrix $A$, the **characteristic equation** of $A$ is the polynomial equation

$$\det(A - rI) = 0.$$

The determinant $\det(A - rI)$ is formed by subtracting $r$ from the diagonal of $A$.
The polynomial $p(r) = \det(A - rI)$ is called the **characteristic polynomial**.

- If $A$ is $2 \times 2$, then $p(r)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(r)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

## Characteristic Equation Examples

Create $\det(\boldsymbol{A} - \boldsymbol{r}\boldsymbol{I})$ by subtracting $\boldsymbol{r}$ from the diagonal of $\boldsymbol{A}$.

Evaluate by the cofactor rule.

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} \end{pmatrix}, \quad \boldsymbol{p}(\boldsymbol{r}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{r} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} - \boldsymbol{r} \end{vmatrix} = (\mathbf{2} - \boldsymbol{r})(\mathbf{4} - \boldsymbol{r})$$

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}, \quad \boldsymbol{p}(\boldsymbol{r}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{r} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} - \boldsymbol{r} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} - \boldsymbol{r} \end{vmatrix} = (\mathbf{2}-\boldsymbol{r})(\mathbf{5}-\boldsymbol{r})(\mathbf{7}-\boldsymbol{r})$$

## Cayley-Hamilton

**Theorem 1 (Cayley-Hamilton)**
A square matrix $\boldsymbol{A}$ satisfies its own characteristic equation.

If $\boldsymbol{p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots a_0}$, then the result is the equation

$$\boldsymbol{(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,}$$

where $\boldsymbol{I}$ is the $\boldsymbol{n \times n}$ identity matrix and $\boldsymbol{0}$ is the $\boldsymbol{n \times n}$ zero matrix.

## The $2 \times 2$ Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the cofactor rule implies

$$p(r) = \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix} = r^2 - (a + d)r + ad - bc.$$

Define

$$\begin{aligned} a_1 &= a + d = \mathbf{trace}(A), \\ a_0 &= ad - bc = \det(A). \end{aligned}$$

Then $p(r) = r^2 + a_1(-r) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ or}$$

$$A^2 - (a + d)A + (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

## Cayley-Hamilton Example

Assume

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}$$

Then

$$\boldsymbol{p}(\boldsymbol{r}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{r} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} - \boldsymbol{r} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} - \boldsymbol{r} \end{vmatrix} = (\mathbf{2} - \boldsymbol{r})(\mathbf{5} - \boldsymbol{r})(\mathbf{7} - \boldsymbol{r})$$

and the Cayley-Hamilton Theorem says that

$$(\mathbf{2}\boldsymbol{I} - \boldsymbol{A})(\mathbf{5}\boldsymbol{I} - \boldsymbol{A})(\mathbf{7}\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$